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The relationship between the Beveridge–Nelson decomposition and other permanent–transitory decompositions that are popular in economics

Kum Hwa Oh [a], Eric Zivot [b,*], Drew Creal [c]

[a] Research Department, The Bank of Korea, Republic of Korea
[b] Department of Economics, University of Washington, United States
[c] Department of Econometrics, VU University Amsterdam, Netherlands

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ABSTRACT

The Beveridge–Nelson (BN) decomposition is a model-based method for decomposing time series into permanent and transitory components. When constructed from an ARIMA model, it is closely related to decompositions based on unobserved components (UC) models with random walk trends and covariance stationary cycles. The decomposition when extended to $I(2)$ models can also be related to non-model-based signal extraction filters such as the HP filter. We show that the BN decomposition provides information on the correlation between the permanent and transitory shocks in a certain class of UC models. The correlation between components is known to determine the smoothed estimates of components from UC models. The BN decomposition can also be used to evaluate the efficacy of alternative methods. We also demonstrate, contrary to popular belief, that the BN decomposition can produce smooth cycles if the reduced form forecasting model is appropriately specified.

1. Introduction

The Beveridge–Nelson (BN) decomposition is a model-based method for decomposing a univariate or multivariate time series into permanent and transitory (PT) components. It defines the stochastic trend as the limiting forecast of the level of the series minus any deterministic components given the current information set. The permanent component is a pure random walk while the remaining movements in the series are the $I(0)$ transitory component. Other than the random walk trend, the BN decomposition does not make assumptions about the structure of the components and the correlations between them. However, it is closely related to decompositions based on unobserved components (UC) models with random walk trends and covariance stationary cycles. The BN decomposition can also be related to non-model-based signal extraction filters such as the Hodrick–Prescott (HP) filter and other Butterworth lowpass filters considered by Gomez (2001). These latter methods are indirectly related to the BN decomposition through their relationships with UC models.

In this paper, we study the BN decomposition when ARIMA models are used as the forecast function. Our contribution is to clarify the relationship between the BN decomposition and other univariate detrending methods popular in economics. In particular, for certain $I(1)$ and $I(2)$ models we describe the relationship between the BN decomposition and UC models with correlated permanent and transitory shocks. We clarify under what conditions the correlation between shocks is identified. For our application to US real GDP, the correlation between components is identified up to a set within the parameter space. If the value of the correlation is within this set, the real-time or filtered estimates of trend and cycle from a UC model will be equivalent to the BN decomposition. We also demonstrate how the BN decomposition can be used as a benchmark to test the over-identifying restrictions that are commonly made in applied macroeconomics research. Examining the over-identifying restrictions can help applied researchers understand the implications that these assumptions have on the decomposition. Many of these results have been stated previously in the literature and part of our contribution is to present these results in a consistent manner.

The BN decomposition holds less relevance for researchers who believe that the trend is not a pure random walk. Consequently, our analysis is limited to models with random walk trend components. There also exist other types of PT decompositions in which the permanent component is an integrated series but not a pure random walk. These include the canonical decomposition of Hillmer and Tiao (1982) and the general PT decompositions of Quah (1992), but these are outside the scope of this paper. As emphasized by Quah (1992), the random walk trend implicit in the BN decomposition maximizes the importance of the permanent

* Corresponding address: Department of Economics, University of Washington, 1100 Campus Parkway Condon Hall Room 401, 98195 Seattle, WA, United States. Tel.: +1 2065436715.
E-mail address: ezivot@u.washington.edu (E. Zivot).

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component. This should always be recognized when interpreting the results of the BN decomposition.

In the following, we begin by presenting the BN decomposition of an $I(1)$ process. Then we discuss the relationship between the BN decomposition for ARIMA(p,1,q) models and different UC models. We extend this to $I(2)$ processes in Section 2. We conclude with an application to US real GDP which emphasizes key points from our previous analysis.

2. The BN decomposition of an $I(1)$ process

Assume that the univariate time series y_t is an $I(1)$ process with Wold representation given by

$$\Delta y_t = \mu + \psi(L)\epsilon_t = \mu + \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j}, \tag{1}$$

where $\Delta = 1 - L$, $\psi(0) = 1$, $\psi(1) \neq 0$, $\sum_{j=0}^{\infty} j^{1/2}|\psi_j| < \infty$, and ϵ_t are iid$(0, \sigma^2)$ one-step-ahead forecast errors. The permanent component or trend τ_t of the BN decomposition of y_t is defined as the limiting forecast minus any deterministic components

$$\tau_t^{BN} = \lim_{J\to\infty} E[y_{t+J} - J\mu|\Omega_t], \tag{2}$$

where Ω_t represents conditioning information available at time t. Writing $y_{t+J} = y_t + \Delta y_{t+1} + \cdots + \Delta y_{t+J}$ and using $E[\Delta y_{t+j}|\Omega_t]$ $(j = 1, \ldots, J)$ based on (1) allow for the analytic evaluation of τ_t^{BN} as

$$\tau_t^{BN} = \mu + \tau_{t-1}^{BN} + \psi(1)\epsilon_t. \tag{3}$$

Hence, the BN trend is a pure random walk with drift μ and has innovation variance $\sigma^2\psi(1)^2$. The transitory component or cycle, c_t^{BN}, is defined as the difference between y_t and the BN trend

$$c_t^{BN} = y_t - \tau_t^{BN} = \tilde{\psi}(L)\epsilon_t, \tag{4}$$

where $\tilde{\psi}(L) = \sum_{j=0}^{\infty} \tilde{\psi}_j L^j$ and $\tilde{\psi}_j = -\sum_{k=j+1}^{\infty} \psi_k$. Solo (1989) showed that the $\frac{1}{2}$-summability of $\psi(L)$ and the uniqueness of the Wold decomposition guarantee the existence and uniqueness of the BN decomposition. From (3) and (4) it is clear that the BN decomposition produces real-time or one-sided estimates of the permanent and transitory components at time t.

An alternative derivation of the BN decomposition follows directly from the factorization $\psi(L) = \psi(1) + (1 - L)\tilde{\psi}(L)$. Then (1) may be rewritten as

$$\Delta y_t = \mu + \psi(1)\epsilon_t + (1 - L)\tilde{\psi}(L)\epsilon_t, \tag{5}$$

which identifies $(\mu + \psi(1)\epsilon_t)/(1 - L)$ as the permanent component and $\tilde{\psi}(L)\epsilon_t$ as the transitory component.

In practice, the BN decomposition can be computed in a number of ways. Typically, it is assumed that Δy_t follows an ARMA(p, q) process so that $\psi(L) = \theta(L)/\phi(L)$ where the orders of $\phi(L)$ and $\theta(L)$ are p and q, respectively, and the roots of $\phi(L) = 0$ and $\theta(L) = 0$ are assumed to lie outside the complex unit circle. A brute force approach is based on estimating an ARMA(p, q) model for Δy_t, using these estimates to compute an estimate of $\psi(1) = \theta(1)/\phi(1)$, and then forming estimates of the components using (2) and (4) with the ARMA residuals in place of ϵ_t. Cuddington and Winters (1987), Miller (1988) and Newbold (1990) provided improvements to this brute force method. These methods are valid if the forecasting model for Δy_t is a univariate ARMA(p, q) model. Ariño and Newbold (1998) extended the algorithm of Newbold (1990) to multivariate forecasting models for Δy_t. Evans and Reichlin (1994) also discussed the BN decomposition for multivariate models. Recently, Morley (2002) provided a very simple state-space approach for calculating the BN decomposition that is valid for any forecasting model for Δy_t that can be cast into state-space form. In particular, suppose $\Delta y_t - \mu$ is a linear combination of the elements of the $m \times 1$ state vector $\boldsymbol{\alpha}_t$

$$\Delta y_t - \mu = \mathbf{z}'\boldsymbol{\alpha}_t,$$

where $\mathbf{z}$ is an $m \times 1$ vector with fixed elements. Suppose further that

$$\boldsymbol{\alpha}_t = \mathbf{T}\boldsymbol{\alpha}_{t-1} + \boldsymbol{\eta}_t, \boldsymbol{\eta}_t \sim \text{iid } N(\mathbf{0}, \mathbf{Q}), \tag{6}$$

such that all of the eigenvalues of $\mathbf{T}$ have modulus less than unity, and $\mathbf{I}_m - \mathbf{T}$ is invertible. Then, Morley (2002) showed that

$$\tau_t^{BN} = y_t + \mathbf{z}'\mathbf{T}(\mathbf{I}_m - \mathbf{T})^{-1}\boldsymbol{\alpha}_{t|t}, \tag{7}$$
$$c_t^{BN} = y_t - \tau_t^{BN} = -\mathbf{z}'\mathbf{T}(\mathbf{I}_m - \mathbf{T})^{-1}\boldsymbol{\alpha}_{t|t},$$

where $\boldsymbol{\alpha}_{t|t} = E[\boldsymbol{\alpha}_t|\Omega_t]$ denotes the filtered or real-time estimate of $\boldsymbol{\alpha}_t$ from the Kalman filter.[1] An important advantage of Morley's approach is its generality. It works the same way for univariate and multivariate forecasting models for Δy_t.

Disadvantages of the methods described above to compute the BN decomposition are that they lose the first observation due to differencing the data, and that they do not provide standard error bands for the extracted trend and cycle estimates. However, as discussed by Morley et al. (2003) (MNZ) and Anderson et al. (2006) and shown in Section 3, the BN decomposition may also be computed directly using the Kalman filter from certain UC models. This allows for the use of all the data and for the calculation of standard error bands for the extracted trend and cycle. This also allows for the extraction of trend and cycle estimates at time t using information in the full sample, Ω_T.

3. The BN decomposition and unobserved components models

The BN decomposition produces a decomposition into permanent and transitory components with minimal assumptions about the structure of the components. The definition of the BN trend (2) identifies the permanent component as a pure random walk, and this result can be used to link the BN decomposition with traditional UC models with random walk trends. The following subsections describe the class of UC models that are consistent with the BN decomposition. Throughout, we assume that Δy_t has a reduced form covariance stationary and invertible ARMA(p, q) representation such that $\psi(L) = \theta(L)/\phi(L)$ in (1).

3.1. *Single-source-of-error model*

The definitions of the BN permanent and transitory components in (3) and (4) suggest the following single-source-of-error (SSOE) state-space representation[2]

$$y_t = \tau_t + c_t, \tag{8}$$
$$(1 - L)\tau_t = \mu + \psi(1)\epsilon_t,$$
$$c_t = \tilde{\psi}(L)\epsilon_t,$$

where $\tilde{\psi}(L)\epsilon_t \sim$ ARMA(p, n) with $n = \max(q - 1, 0)$. It is clear from (8) that the innovations to the permanent and transitory components are perfectly correlated

$$\rho = \frac{\text{cov}(\psi(1)\epsilon_t, \tilde{\psi}(0)\epsilon_t)}{\sqrt{\text{var}(\psi(1)\epsilon_t)\text{var}(\tilde{\psi}(0)\epsilon_t)}} = \frac{\psi(1)\tilde{\psi}(0)}{|\psi(1)\tilde{\psi}(0)|} = -1 \text{ or } 1,$$

where the sign of ρ depends on the sign of $\tilde{\psi}(0)$. Hence, there always exists a UC representation with perfectly correlated shocks that is consistent with the BN decomposition. However, as discussed by MNZ, Eq. (8) is not the only UC representation that is consistent with the BN decomposition. We note that Ord et al. (1997) advocated the use of SSOE UC models because they do

[1] Throughout the paper we refer to filtered estimates as real-time estimates based on information only available at time t, and smoothed estimates as final estimates based on all available sample information.

[2] Anderson et al. (2006) gave a slightly different, but equivalent, formulation of the SSOE model that includes ϵ_t in the measurement equation.