Computers and Structures 145 (2014) 12–22

Contents lists available at ScienceDirect

Computers and Structures

journal homepage: www.elsevier.com/locate/compstruc

A self-regularized approach for deriving the free–free flexibility and stiffness matrices

Jeng-Tzong Chen [a,b,*], Wen-Sheng Huang [a], Jia-Wei Lee [a], Ya-Ching Tu [a]

[a] Department of Harbor and River Engineering, National Taiwan Ocean University, Taiwan
[b] Department of Mechanical and Mechatronic Engineering, National Taiwan Ocean University, Taiwan

ARTICLE INFO

Article history:
Received 20 April 2014
Accepted 31 July 2014
Available online 19 September 2014

Keywords:
Free–free flexibility matrix
Free–free stiffness matrix
Rigid body modes
Spurious force modes
Fichera's method
Generalized inverse

ABSTRACT

Motivated by Fichera's idea for regularizing the rank-deficiency model, we derive the free–free flexibility matrices by inverting the bordered stiffness matrix. The singular stiffness matrix of a free–free structure is expanded to a bordered matrix by adding n slack variables, where n is the nullity of the singular stiffness matrix. Besides, the corresponding n constraints are accompanied to result in a nonsingular matrix. The constraints filter out the homogeneous solution for the regularized solution. By inverting the nonsingular matrix, we can obtain the free–free flexibility matrix from the submatrices. The value of the extra degree of freedom shows the role of no solution (nonzero case) or infinite solution (zero case) with respect to the loading vector. After constructing the bordered system, the equilibrium of the specified force and the compatibility of the specified displacement can be tested according the zero slack variable. Similarly, the free–free flexibility matrix is obtained from the free–free stiffness matrix. Finally, four examples, a rod with symmetric stiffness, a plane truss, a beam and a bar with unsymmetric stiffness, were demonstrated to see the validity of the present formulation.

1. Introduction

There are two kinds of rank-deficiency problems in the boundary element method (BEM) or finite element method (FEM). Physically speaking, a rigid body mode exists in a free–free structure for structural mechanics. This means that the free–free stiffness matrix is singular in companion with zero eigenvalues (singular values). No matter which numerical method, BEM or FEM, is employed, the obtained influence matrix (stiffness matrix) is rank deficient. Such outcome occurs naturally in the Neumann problem or the traction problem for potential and elasticity problems, respectively [1–7].

Regarding the Dirichlet problem in potential theory or constrained structure in elasticity, the solution is mathematically unique. However, this yields rank-deficiency problem if a single-layer potential approach (indirect BEM) is employed to solve it for a critical scale (degenerate scale). To avoid this unreasonable model, Fichera proposed a well-posed model to make it full rank [8]. Two steps are utilized at the same time. One is to introduce a free constant field. The other is to provide a corresponding constraint. After discretization, the singular system is transformed to a nonsingular bordered system. It is interesting to find that the discretization system to promote the full rank is the same as the self-regularized linear algebraic system for deriving the flexibility of a free–free body. Following this finding, we will drive the free–free flexibility in the way of inverting the full-rank bordered matrix. On the contrary, finding the free–free stiffness from the free–free flexibility is also discussed. Physical rigid-body modes for the displacement as well as nonphysical spurious force modes corresponding to zero singular values are found. The spurious mode also appears in the finite element method. For example, hourglass mode occurs in the reduced integration to soften the shear locking. This zero-energy mode is not physically realizable but due to mathematics. The nonphysical outcome due to mathematics (rank deficiency) needs regularization in the mathematical model. Table 1 indicates the relation between mathematics and structural mechanics. Zero eigenvalues imply the rigid body mode (physics) and spurious mode (mathematics). Bordered matrix introduces an extra degree of freedom and transforms a singular matrix to be a nonsingular one. Free–free structure yields a rank-deficiency matrix.

Regarding the inverse of a singular matrix, Felippa et al. [1] have introduced the dual of free–free stiffness **K** and flexibility **F**. They also emphasized the potential applications of free–free flexibility for substructure-based solution algorithm in the direct flexibility matrix. Construction of free–free flexibility matrices can be derived by using the generalized inverse of stiffness. Derivation

* Corresponding author at: Department of Harbor and River Engineering, National Taiwan Ocean University, Taiwan.
E-mail address: jtchen@mail.ntou.edu.tw (J.-T. Chen).

http://dx.doi.org/10.1016/j.compstruc.2014.07.024
0045-7949/

Table 1
Relation between mathematics and structural mechanics.

Mathematics	Structural mechanics
Null space	Spurious mode
	Rigid body mode
Rank deficiency	Free–free structure
Bordered matrix	Adding an extra degree of freedom
Generalized inverse	Free–free flexibility matrix
Moore–Penrose	Free–free stiffness matrix
Influence matrix	Stiffness or flexibility matrix

of flexibility and stiffness matrices of rod and beam was also investigated by using the dual BEM [6]. Generalized inverse has been studied by Fredholm, Moore and Penrose in the twenty century. Generalized inverses was mathematically studied by using the bordered matrix [9]. However, its engineering applications in structural mechanics were not noticed in that book. In this paper, the proposed self-regularized approach is similar to the Moore–Penrose/Singular Value Decomposition (SVD) approach for computing pseudo-inverses of rank-deficient matrices. But the main difference is that we add a slack variable and a corresponding constraint in the present method. This idea was similarly used in the optimization theory by adding n slack variables. Besides, the flexibility matrix is more efficient in the substructure method, especially in the case of replacement of failure element. The idea was addressed in the Felippa's paper [10].

In non-linear geometry analysis, the arc length method was introduced in the analysis which is similar to the present slack variable. Introducing a slack variable is very popular to transform an inequality to an equation in the optimization theory. The same algorithm in this article is the addition of one degree of freedom in accompany with an extra equation. An incremental force parallel to the critical eigenvectors of the tangential stiffness matrix is separately treated. Using eigenvector projections, we can improve convergence in non-linear finite element equilibrium iterations [11,12]. Although with other objectives in [13], very similar analysis and derivation methods were used in the context of 'eigenvector projections', in the stabilization of non-linear equilibrium iteration methods already in the 1980s. In those, the parts of an incremental force parallel to the critical eigenvectors of the tangential stiffness matrix are separately treated, which is a very similar idea as in this paper. The treatment distinguishes and separately handles two components of the 'load vector', and thereby also the 'displacement response': one parallel to the critical/singular directions, and one orthogonal part. This is a fundamental fact of structural response, which is not related to the need to invert the structural stiffness matrix or parts of it. It also gives a way to explain the introduced unknown coefficients. It corresponds to the value c in this paper. Eigenvector projection provides an efficient way to improve the stability in the iterations for the choice of the optimal corrections. Checking eigenvectors corresponding to near-zero eigenvalues is very important for selecting the damping. In our approach, the singular vector of corresponding to zero singular value provides us the row and column vectors in the bordered matrices, where the unknown coefficient c is introduced.

Based on the structures with symmetry, group-theoretical insight and graph theory can decompose the system to a small one and bypass intrinsic singularities. Related works can be found in the four references [14–17]. However, our approach introduces a slack variable as well as a corresponding constraint to deal with rank-deficient matrices.

In this paper, we derive the free–free flexibility matrix directly from the physical concept as well as the mathematical technique of bordered matrix in the linear algebra. Four examples, a rod with symmetric stiffness, a plane truss, a beam and a bar with unsymmetric stiffness, were demonstrated to see the validity of the present formulation.

2. Formulation

In potential theory, the single-layer representation model is often used to solve the boundary value problem as shown below:

$$u(x) = \int_B U(x,s)\phi(s)dB(s), \quad x \in D, \tag{1}$$

where $u(x)$ is the potential field, $\phi(s)$ is the unknown boundary density, $U(x,s)$ is the fundamental solution and B is the boundary of the domain D.

However, Eq. (1) may fail for the Dirichlet problem with a specific scale (degenerate scale). To overcome this ill-posed (rank-deficiency) model, Fichera proposed a regularized formulation by simultaneously adding a constant and an extra constraint as shown below:

$$u(x) = \int_B U(x,s)\phi_r(s)dB(s) + c, \quad x \in D, \tag{2}$$

$$\int_B \phi_r(s)dB(s) = 0, \quad s \in B. \tag{3}$$

After discretizing the boundary by using the constant element, Eq. (1) reduces to

$$\mathbf{U}\underset{\sim}{\phi} = \underset{\sim}{b}. \tag{4}$$

By employing the boundary element implementation, Eqs. (2) and (3) together yield

$$\begin{bmatrix} \mathbf{U} & \{1\} \\ \{l\} & 0 \end{bmatrix} \begin{Bmatrix} \underset{\sim}{\phi_r} \\ c \end{Bmatrix} = \begin{Bmatrix} \underset{\sim}{b} \\ 0 \end{Bmatrix}, \tag{5}$$

where $\mathbf{U}$ is the influence matrix and $\{l\}$ is the vector of length for boundary elements. It is noted that $\underset{\sim}{\phi}$ in Eq. (4) is the unregularized unknown vector, while ϕ_r in Eq. (5) is the regularized unknown vector.

By using analogy between the singular stiffness matrix for structural mechanics and the influence matrix for the indirect BEM as shown in Fig. 1, a regularized (bordered) matrix provides an alternative way to construct the free–free flexibility matrix.

The linear algebraic system is

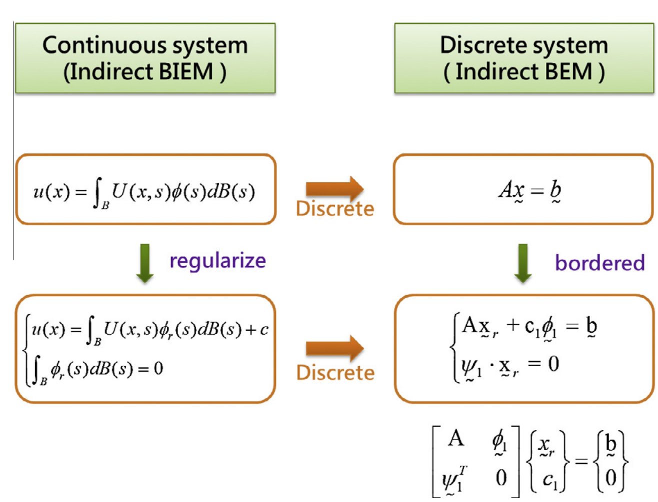

Fig. 1. The self-regularized linear algebraic system from the continuous BIE system.