Computers and Structures 142 (2014) 1–14

Contents lists available at ScienceDirect

Computers and Structures

journal homepage: www.elsevier.com/locate/compstruc

The method of finite spheres for wave propagation problems

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ARTICLE INFO

Article history:
Received 26 March 2014
Accepted 27 May 2014
Available online 19 July 2014

Keywords:
Method of finite spheres
Meshless method
Wave propagation
Harmonic functions
Numerical integration
Implicit time integration

ABSTRACT

We present in this paper the development and use of a novel and promising numerical scheme, the method of finite spheres, for the analysis of wave propagations. The solution of two-dimensional linear elastic and visco-elastic waves is considered. The procedure does not require a mesh and hence avoids element distortions. We discuss the construction of the interpolations in which harmonic functions are included to make the method effective for the analysis of wave propagations. A simple and efficient numerical integration scheme is proposed and the solution effort is evaluated versus using the standard finite element method. Several numerical examples are used to demonstrate capabilities of the method.

1. Introduction

The objective in the development of the method of finite spheres is to obtain a reliable and efficient method to solve complex boundary value problems without the construction of a mesh [1–6]. The method was developed during a period of increasing awareness and promise of meshless methods with the aim to overcome some inherent limitations of numerical methods that rely on the use of a mesh [7]. The finite element method requires the discretization of a problem domain into a set of non-overlapping elements. The consequence of a poorly constructed discretization can be a significant loss in accuracy of the solution [8]. In addition, for wave propagation problems, the standard finite element method may not be effective since very fine meshes are required and even when using such fine meshes the solution may still show significant spurious oscillations, and dispersion and dissipation errors. The enriched finite element method using an implicit time integration scheme was developed to address these difficulties [9–11]. However, measured on the experiences obtained, there is still significant need for improved solution procedures.

Several other methods have been proposed for the solution of wave propagation problems [12–32]. The spectral element method is a high-order Lagrangian-based finite element technique that combines the approach of using finite elements with the accuracy of spectral methods. Lagrange polynomials approximate the field variables and Gauss–Lobatto quadrature is used for the required integrations leading to a diagonal mass matrix when using quadrilateral or hexahedral elements in two- and three-dimensional solutions, respectively, which is advantageous for the analysis of transient wave propagation problems [13–15]. The major difficulty, however, is to mesh complex two- and three-dimensional domains and obtain effective solutions using a mixture of elements.

Meshless methods have attracted significant attention for the analyst to solve boundary value problems without the use of a predefined mesh. Some meshless procedures have also been developed for the solution of wave propagation problems and can generally be categorized into strong-form and weak-form methods. Smoothed particle hydrodynamics (SPH) is a well-known strong-form method, initially used for modeling astrophysical phenomena [16,17]. Weak-form methods include the element-free Galerkin method (EFG) [18,19], the meshless local Petrov–Galerkin method (MLPG) [20–23], and the method of finite spheres (MFS) [1]. Meshless methods can also be categorized by their choice of interpolation functions. For example, methods employing radial basis functions (RBF) have been developed for the solution of transient acoustic wave propagation problems [24–26].

The SPH scheme is a commonly used method for simulating fluids. The method represents the field quantity in an integral form based on kernel approximation functions. Current research has also illustrated the applicability of SPH for the solution of wave propagation problems in solid mechanics. The major difficulties commonly cited for SPH are tensile instability and boundary deficiency [16,17]. Tensile instability refers to an unstable solution when tensile stresses are present. Boundary deficiency is a consequence of not satisfying zeroth-order consistency near or on the boundary of the problem domain. Improvements addressing these

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http://dx.doi.org/10.1016/j.compstruc.2014.05.012
0045-7949/

difficulties have been developed and the improved schemes were shown to provide good accuracy for the solution of shock wave propagations in viscoplastic media [17]. However, the SPH scheme still requires the use of a large number of nodes to obtain reasonably accurate solutions and the use of some 'adjustable solution factors' that render the method not applicable, for example, to the solution of simple linear elastic static problems.

The element-free Galerkin method has been shown to provide good results for acoustic and elastic wave propagation problems [19]. Imposing the Dirichlet boundary conditions, however, is not straightforward since the interpolation functions do not satisfy the Kronecker delta property. Lagrange multipliers can be used, but this leads to a larger system of algebraic equations and a loss of positive definiteness which reduces the computational efficiency. Alternatively, the penalty method can be employed, but then appropriate penalty factors need to be chosen, that may depend on the problem considered [18,19].

The meshless local Petrov–Galerkin method adopts trial and test functions from different approximation spaces, resulting in various formulations [20–23]. The MLPG approach has been used to solve for the propagation and scattering of electromagnetic waves, where the trial functions are constructed from moving least squares approximations and the test functions from solutions of Green's problem [22]. The method has also been used for the solution of wave propagations in three-dimensional poroelastic solids, with the trial functions constructed using radial basis functions and the test functions being simply unit step functions [23]. While the given numerical solutions are in good agreement with analytical solutions, these methods are computationally expensive due to the non-symmetry of the coefficient matrices and the numerical integration of complex expressions within the subdomains.

The method of finite spheres is related to the above techniques in that it is based on a weak formulation of boundary value problems using overlapping subdomains, and hence also does not require a mesh. In fact, the only difference to the standard finite element method is that the spheres (subdomains, or elements) overlap, and indeed other subdomains (like bricks) could be employed. The method is using symmetric coefficient matrices and was initially presented for the analysis of linear static solids and fluids [1]. Further research focused on the method of finite spheres in a mixed formulation, improved numerical integration, automatic discretization, coupling with the finite element method, and enrichment strategies [2–6]. In this paper, we seek to demonstrate that there is significant promise of reliability and efficiency for the method of finite spheres in the solution of wave propagation problems.

We briefly review the method of finite spheres in Section 2. We give the interpolation functions used for general elliptic problems and introduce the special interpolation functions used for wave propagation problems. In Section 3, the formulation for the analysis of two-dimensional linear elastic wave propagation problems is presented. Since efficiency is a particular concern for the use of the method of finite spheres, a simple improved numerical integration scheme is given in Section 4. Finally, to illustrate the capabilities of the developed scheme, we present in Section 5 the results of several wave propagation problems in elastic and visco-elastic media.

Since we consider in this paper only the solution of two-dimensional problems, we should note that when referring to a "sphere", a "disk" is implied.

2. Interpolation scheme

The method of finite spheres uses the Shepard partition of unity functions. The interpolations are defined by the Shepard functions times local basis functions that can include any desired enrichments [1]. Consistency and continuity conditions are satisfied by proper selection of the local basis functions that together span the local approximation space. Computational efficiency depends on the suitability of these functions when considering the degree of the governing partial differential equations and the nature of the solutions to be predicted.

2.1. Shepard partition of unity functions

Let $V \in R^d (d = 1, 2, or\ 3)$ be an open bounded domain and let S be the domain boundary, with $S = S_u \cup S_f$ and $S_u \cap S_f = 0$, where S_u is the Dirichlet boundary and S_f is the Neumann boundary. Then let $\{B(\underline{x}_I, r_I);\ I = 1, 2, \ldots, N\}$ be a set of spheres which form a covering for V, i.e., $V \subset \bigcup_{I=1}^{N} B(\underline{x}_I, r_I)$, where $\underline{x}_I$ and r_I refer to the center and radius of the sphere B_I, respectively, and where I is the nodal label of each sphere and N is the total number of spheres. The unit normal to the domain boundary, $\underline{n}$, is positive in the outward direction. As illustrated in Fig. 1, spheres are either interior spheres, entirely within the problem domain, or boundary spheres intercepting the domain boundary.

Let $W_I(\underline{x})$ denote a positive radial weighting function of the form $W_I(\underline{x}) = W(s_I)$, with $s_I = ||\underline{x} - \underline{x}_I||/r_I$ where $||\cdot||$ is the Euclidean norm. We use the quartic spline weighting function defined as

$$W(s_I) = \begin{cases} 1 - 6s_I^2 + 8s_I^3 - 3s_I^4, & 0 \leqslant s_I \leqslant 1 \\ 0, & s_I > 1 \end{cases} \tag{1}$$

The Shepard partition of unity function is then given by

$$\varphi_I^0(\underline{x}) = \frac{W_I}{\sum_{J=1}^{N} W_J}, \quad I = 1, 2, \ldots, N \tag{2}$$

Hence $\sum_{I=1}^{N} \varphi_I^0(\underline{x}) = 1\ \forall \underline{x} \in V$. These functions are rational, non-polynomial functions satisfying zeroth order consistency, ensuring that rigid body modes can be reproduced exactly.

To generate approximation spaces of higher order consistency, a local approximation space $V_I^h = \text{span}_{m \in \mathcal{J}}\{p_m(\underline{x})\}$ is defined at each node I, where h is a measure of the sphere size, $\mathcal{J}$ is an index set, and $p_m(\underline{x})$ is a member of the local basis. Then the global approximation space V_h is defined as the product of the Shepard function at each node I and the functions from the local bases

$$V_h = \sum_{I=1}^{N} \varphi_I^0 V_I^h \tag{3}$$

Hence any function v_h in the solution space V_h can be written as

$$v_h(\underline{x}) = \sum_{I=1}^{N} \sum_{m \in \mathcal{J}} h_{Im}(\underline{x}) \alpha_{Im} \tag{4}$$

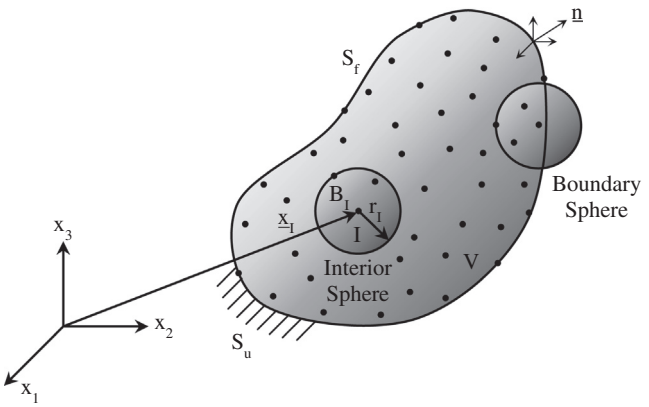

Fig. 1. General problem domain V with domain boundary $S = S_u \cup S_f$.