



Parameter estimation and bias correction for diffusion processes

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ABSTRACT

This paper considers parameter estimation for continuous-time diffusion processes which are commonly used to model dynamics of financial securities including interest rates. To understand why the drift parameters are more difficult to estimate than the diffusion parameter, as observed in previous studies, we first develop expansions for the bias and variance of parameter estimators for two of the most employed interest rate processes, Vasicek and CIR processes. Then, we study the first order approximate maximum likelihood estimator for linear drift processes. A parametric bootstrap procedure is proposed to correct bias for general diffusion processes with a theoretical justification. Simulation studies confirm the theoretical findings and show that the bootstrap proposal can effectively reduce both the bias and the mean square error of parameter estimates, for both univariate and multivariate processes. The advantages of using more accurate parameter estimators when calculating various option prices in finance are demonstrated by an empirical study.

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1. Introduction

Diffusion processes have been commonly used in finance to model stochastic dynamics of financial securities following the works of Black and Scholes (1973) and Merton (1973) which established the foundation of option pricing theory in finance. There has been phenomenal growth in financial products and instruments powered by these processes, as documented in Sundaresan (2000). A d -dimensional parametric diffusion process $\{X_t \in \mathcal{R}^d; t \geq 0\}$ is defined by the following stochastic differential equation

$$dX_t = \mu(X_t; \theta)dt + \sigma(X_t; \theta)dB_t, \quad (1.1)$$

where θ is a q -dimensional parameter, $\mu(\cdot; \theta) : \mathcal{R}^d \rightarrow \mathcal{R}^d$ and $\sigma(\cdot; \theta) = (\sigma_{ij})_{d \times p} > 0 : \mathcal{R}^d \rightarrow \mathcal{R}^{d \times p}$ are drift and diffusion functions representing, respectively, the conditional mean and variance of the infinitesimal change of X_t at time t , and B_t is a p -dimensional Brownian motion. The existence and uniqueness of the process $\{X_t; t \geq 0\}$ satisfying (1.1) and its probability properties are given in Stroock and Varadhan (1979).

A unique feature of statistical inference for diffusion processes is that, despite these processes being continuous-time stochastic models, their observations are made only at discrete time points, say at n equally spaced $\{t\delta\}_{t=0}^n$. Here δ is the sampling interval and can be either fixed or very small corresponding to high-frequency data. See Lo (1988), Bibby and Sørensen (1995), Ait-Sahalia (2002), Ait-Sahalia and Mykland (2003), and Fan (2005) for discussions and overviews for estimation of diffusion processes, based on discrete observations.

Short-term interest rates are fundamental quantities in finance, as they define excess asset returns and risk premiums of other assets and their derivative prices. A family of diffusion processes for the interest rates dynamics, consists of the following linear drift processes

$$dX_t = \kappa(\alpha - X_t)dt + \sigma(X_t, \psi)dB_t, \quad (1.2)$$

where α , κ and ψ are unknown parameters. The linear drift prescribes a mean-reversion of X_t toward the long term mean α at a speed κ . The diffusion function $\sigma(X_t, \psi)$ can accommodate a range of patterns in volatility. A sub-family of the above linear drift processes is given by assigning $\sigma(X_t, \psi) = \sigma X_t^\rho$ with $\psi = (\sigma^2, \rho)$. Important members of this sub-family are the Vasicek model (Vasicek, 1977) with $\rho = 0$ and the CIR model (Cox et al., 1985) with $\rho = 1/2$. Both Vasicek and CIR models are commonly used in finance due to (i) both having simple and

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attractive financial interpretations; and (ii) both admitting close-form solutions. The latter facilitates explicit calculations of various option prices.

Despite the critical roles played by these interest rate processes, it is well known, empirically, that estimation of the drift parameters κ and α can incur large bias and/or variability; see for instance Ball and Torous (1996) and Yu and Phillips (2001). Merton (1980) had earlier discovered the difficulty with the drift parameter estimation for the Black–Scholes diffusion model, where both the drift and diffusion functions are constant functions. The difficulty with the drift parameter estimation is encountered by virtually all the estimation approaches, including the maximum likelihood estimation. The problem increases when the process has a lack of dynamics, which happens when κ is small. Indeed, as reported in Phillips and Yu (2005) and our simulation study, the maximum likelihood estimator for κ can incur relative bias of more than 200% even the processes are observed monthly for more than 10 years. This is serious, as poor qualitative estimates can produce severely biased option prices.

The objectives of this paper are (i) to understand the above empirical phenomena by developing expansions to the bias and variance of estimators for the Vasicek, CIR and general linear drift diffusion processes; and (ii) to propose a bias correction approach that is applicable to general diffusion processes. Two asymptotic regimes are considered in our analysis. One has δ (the sampling interval) fixed while the sample size $n \rightarrow \infty$. The other has δ converges to zero as $n \rightarrow \infty$. The latter corresponds to high frequency data, and allows simplification of results as compared to results for the fixed- δ case.

The bias and variance expansions reveal that regardless of whether δ is fixed or diminishing to zero, the bias of the κ estimators and the variances of the two drift parameters estimators are effectively at the order of $(n\delta)^{-1}$, depending on the amount of time over which the process is observed. Our analysis also reveals that the bias and variance of the estimators for the diffusion parameter σ^2 basically enjoys much smaller orders at n^{-1} . These explain why estimation of κ incurs more bias than the other parameters, and why the drift parameter (κ and α) estimates are more variable than those of the diffusion parameter.

We then propose a parametric bootstrap procedure for bias correction for general diffusion processes. Both theoretical and empirical analysis show that the proposed bias correction effectively reduces the bias without inflating the variance. The proposed bootstrap procedure can be combined with a range of parameter estimators, including the approximate likelihood estimation of Ait-Sahalia (2002, 2008).

The paper is structured as follows. Section 2 outlines parameter estimators used in our analysis. Bias and variance expansions for the estimators of Vasicek, CIR and the linear drift processes are presented in Section 3. Section 4 discusses the bootstrap bias correction. Simulation results are reported in Section 5. Section 6 analyzes a dataset of Fed fund rates. All technical details are deferred to the Appendix.

2. Parameter estimation for diffusion processes

2.1. A general overview

Let $X_0, X_\delta, \dots, X_{n\delta}$ be discrete observations from process (1.1) at equally spaced time points $\{t\delta\}_{t=0}^n$ over a time interval $[0, T]$ where $T = n\delta$. To simplify notation, we write these observations as $\{X_t\}_{t=0}^n$ by hiding δ whenever doing so does not lead to confusion. As a diffusion process is Markovian, the maximum likelihood estimation (MLE) is the natural choice for parameter estimation if its transitional density is known. However, for most diffusion processes, their transitional distributions are not explicitly known,

which prevents the use of the MLE. In these cases, several methods are available, which include the martingale estimating equation approach of Bibby and Sørensen (1995); the pseudo-Gaussian likelihood approach of Nowman (1997); the Generalized Method of Moments (GMM) estimator of Hansen and Scheinkman (1995); the Efficient Method of Moments of Gallant and Tauchen (1996) and the approximate likelihood approach of Ait-Sahalia (2002). Ait-Sahalia and Mykland (2003, 2004) consider likelihood and the GMM based estimation when δ is random. Nonparametric estimators for the drift and diffusion functions have been also proposed; see Fan (2005) for reviews.

We carry out our analysis under two asymptotic regimes. It is assumed, in the first regime, that $n \rightarrow \infty$ while δ is a fixed constant; and in the second regime that

$$n \rightarrow \infty, \delta \rightarrow 0, T = n\delta \rightarrow \infty \quad (2.1)$$

and for some $k > 2$ $T\delta^{1/k} \rightarrow \infty$.

In the second regime, δ diminishes to zero while the total observational time goes to infinity as $n \rightarrow \infty$. The last part of (2.1) is used to bound remainder terms in moment expansions. We note that $T \rightarrow \infty$ mimics the standard asymptotic of $n \rightarrow \infty$ and, as shown in our analysis, is the main driving force in determining the bias and variance properties in the drift parameter estimation.

The motivations for assuming $\delta \rightarrow 0$ besides $n \rightarrow \infty$ are two-fold. One is that high frequency financial data are increasingly available. Another is to accommodate discretization based estimators, which normally requires $\delta \rightarrow 0$ to make the discretization error diminish to zero fast enough so that the estimators are consistent.

2.2. Estimation for Vasicek process

The Vasicek process satisfies the univariate stochastic differential equation

$$dX_t = \kappa(\alpha - X_t)dt + \sigma dB_t. \quad (2.2)$$

It is the Ornstein–Uhlenbeck process and was proposed by Vasicek (1977) for interest rate dynamics. The conditional distribution of X_t given X_{t-1} is

$$X_t | X_{t-1} \sim N \left\{ X_{t-1}e^{-\kappa\delta} + \alpha(1 - e^{-\kappa\delta}), \frac{1}{2}\sigma^2\kappa^{-1}(1 - e^{-2\kappa\delta}) \right\}$$

and the stationary distribution is $N(\alpha, \frac{1}{2}\sigma^2\kappa^{-1})$. The conditional mean and variance of X_t given X_{t-1} are

$$E(X_t | X_{t-1}) = X_{t-1}e^{-\kappa\delta} + \alpha(1 - e^{-\kappa\delta}) =: \mu(X_{t-1}) \quad \text{and} \quad (2.3)$$

$$\text{Var}(X_t | X_{t-1}) = \frac{1}{2}\sigma^2\kappa^{-1}(1 - e^{-2\kappa\delta}). \quad (2.4)$$

Let $\phi(x)$ be the density function of the standard normal distribution $N(0, 1)$. Then, the likelihood function of $\theta = (\kappa, \alpha, \sigma^2)$ is

$$L(\theta) = \phi \left(\sigma^{-1} \sqrt{2\kappa} (X_0 - \alpha) \right) \times \prod_{t=1}^n \phi \left(\sigma^{-1} \sqrt{2\kappa(1 - e^{-2\kappa\delta})^{-1}} \{X_t - \mu(X_{t-1})\} \right).$$

Ignoring the first component in $L(\theta)$ involving X_0 , the maximum likelihood estimators (MLE) can be obtained explicitly by

$$\hat{\kappa} = -\delta^{-1} \log(\hat{\beta}_1), \quad \hat{\alpha} = \hat{\beta}_2 \quad \text{and} \quad \hat{\sigma}^2 = 2\hat{\kappa}\hat{\beta}_3(1 - \hat{\beta}_1^2)^{-1} \quad (2.5)$$

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