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The limit distribution of the estimates in cointegrated regression models with multiple structural changes[☆]

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ABSTRACT

We study estimation and inference in cointegrated regression models with multiple structural changes allowing both stationary and integrated regressors. Both pure and partial structural change models are analyzed. We derive the consistency, rate of convergence and the limit distribution of the estimated break fractions. Our technical conditions are considerably less restrictive than those in Bai et al. [Bai, J., Lumsdaine, R.L., Stock, J.H., 1998. Testing for and dating breaks in multivariate time series. Review of Economic Studies 65, 395–432] who considered the single break case in a multi-equations system, and permit a wide class of practically relevant models. Our analysis is, however, restricted to a single equation framework. We show that if the coefficients of the integrated regressors are allowed to change, the estimated break fractions are asymptotically dependent so that confidence intervals need to be constructed jointly. If, however, only the intercept and/or the coefficients of the stationary regressors are allowed to change, the estimates of the break dates are asymptotically independent as in the stationary case analyzed by Bai and Perron [Bai, J., Perron, P., 1998. Estimating and testing linear models with multiple structural changes. Econometrica 66, 47–78]. We also show that our results remain valid, under very weak conditions, when the potential endogeneity of the non-stationary regressors is accounted for via an increasing sequence of leads and lags of their first-differences as additional regressors. Simulation evidence is presented to assess the adequacy of the asymptotic approximations in finite samples.

1. Introduction

Issues related to structural change have received considerable attention in the statistics and econometrics literature (see Perron (2006), for a survey). In the last fifteen years or so, substantial advances have been made in the econometrics literature to cover models at a level of generality that allows a host of interesting practical applications in the context of unknown change points. These include models with general stationary regressors and errors that can exhibit temporal dependence and heteroskedasticity. Andrews (1993) and Andrews and Ploberger (1994) provide a comprehensive treatment of the problem of testing for structural change assuming that the change point is unknown. Bai (1997) studies the least squares estimation of a single change point in regressions involving stationary and/or trending regressors. He derives the consistency, rate of convergence and the limiting distribution of the change point estimator under general conditions on the regressors and the errors. Bai and Perron (1998) extend the testing and estimation analysis to the case of multiple structural changes, while Bai and Perron (2003) present an efficient algorithm to obtain the break dates corresponding to the global minimizers of the sum of squared residuals. Perron and Qu (2006) consider the case in which restrictions within or across regimes are imposed. Qu and Perron (2007) cover the more general case of multiple structural changes in a system of equations allowing arbitrary restrictions on the parameters.

When dealing with non-stationary variables, the literature is less extensive. With respect to testing, Hansen (1992b) develops tests of the null hypothesis of no change in models where all coefficients are allowed to change. An extension to partial changes has been analyzed by Kuo (1998). The tests considered are the Sup and Mean LM tests directed against an alternative of a one time change in parameters. Hao (1996) also suggests the use of the exponential LM test. Seo (1998) considers the Sup, Mean and Exp versions of the LM test within a cointegrated VAR setup. The Sup and Mean LM tests in this setup are shown to have a similar asymptotic distribution as the Sup and Mean LM tests of Hansen (1992b). Kejriwal and Perron (2008) show that such tests can suffer

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from important lack of power in finite samples and be subject to a non-monotonic power function such that the power decreases as the magnitude of the break increases. They suggest modified Sup–Wald type tests that perform considerably better.

With respect to estimation, Perron and Zhu (2005) analyze the properties of parameter estimates in models where the trend function exhibits a slope change at an unknown date and the errors can be either stationary or have a unit root. With integrated variables, the case of most interest is that of a framework in which the variables are cointegrated. Accounting for parameter shifts is crucial in cointegration analysis since it normally involves long spans of data which are more likely to be affected by structural breaks. The goal is then to test whether the cointegrating relationship has changed and to estimate the break dates and form confidence intervals for them. In this respect, an important paper is that of Bai et al. (1998) who consider a single break in a multi-equations system and show the estimates obtained by maximizing the likelihood function to be consistent. They also obtain a limit distribution of the estimate of the break date under a shrinking shift scenario assuming that the coefficients associated with the trend and the non-stationary regressors shrink faster than those pertaining to the stationary regressors.

The aim of this paper is to provide a comprehensive treatment of issues related to estimation and inference with multiple structural changes, occurring at unknown dates, in cointegrated regression models. Our work builds on that of Bai and Perron (1998) who undertake a similar treatment in a stationary framework. Our framework is general enough to allow both stationary and non-stationary variables in the regression. The assumptions regarding the distribution of the error processes are mild enough to allow for general forms of serial correlation and conditional heteroskedasticity, as well as mild forms of unconditional heteroskedasticity. Moreover, we analyze both pure and partial structural change models. A partial change model is useful in allowing potential savings in the number of degrees of freedom, an issue particularly relevant for multiple changes. It is also important in empirical work since it helps to isolate the variables which are responsible for the failure of the null hypothesis. The parameter estimates of the regression coefficients and the break dates are obtained by minimizing the sum of squared residuals. We derive the consistency, rate of convergence and limiting distribution of the estimated break fractions under much weaker conditions than those in Bai et al. (1998). We show that if the coefficients of the integrated regressors are allowed to change, the estimated break fractions are asymptotically dependent so that confidence intervals need to be constructed jointly. Methods to construct such confidence intervals are discussed. If, however, only the intercept and/or the coefficients of the stationary regressors are allowed to change, the estimates of the break dates are asymptotically independent as in the stationary framework analyzed by Bai and Perron (1998). Though our theoretical results hold under much weaker conditions than those of Bai et al. (1998) and allow for multiple breaks, our analysis is restricted to a single cointegrating vector unlike theirs which is valid in a multi-equations system which, thereby allows multiple cointegrating vectors. In the multiple break case, the fact that the estimated break fractions are asymptotically dependent complicates the analysis considerably and the extension to a multi-equations system is outside the scope of this paper.

This article is organized as follows. Section 2 presents the model and assumptions. In Section 3, we derive the consistency, rate of convergence and limiting distribution of the estimates of the break dates. Section 4 presents the results of simulation experiments to assess the adequacy of the asymptotic approximations in finite samples. Section 5 offers concluding remarks and all technical derivations are included in a mathematical Appendix.

2. The model and assumptions

Consider the following linear regression model with m breaks ($m+1$ regimes):

$$y_t = c_j + z'_{ft}\delta_f + z'_{bt}\delta_{bj} + x'_{ft}\beta_f + x'_{bt}\beta_{bj} + u_t \quad (t = T_{j-1}+1, \ldots, T_j) \tag{1}$$

for $j = 1, \ldots, m+1$, where $T_0 = 0$, $T_{m+1} = T$ and T is the sample size. In this model, $x_{ft}\,(p_f \times 1)$ and $x_{bt}\,(p_b \times 1)$ are vectors of $I(0)$ variables while $z_{ft}\,(q_f \times 1)$ and $z_{bt}\,(q_b \times 1)$ are vectors of $I(1)$ variables defined by

$$z_{ft} = z_{f,t-1} + u^f_{zt}$$
$$z_{bt} = z_{b,t-1} + u^b_{zt}$$
$$x_{ft} = \mu_f + u^f_{xt}$$
$$x_{bt} = \mu_b + u^b_{xt}$$

where z_{f0} and z_{b0} are assumed, for simplicity, to be either $O_p(1)$ random variables or fixed finite constants. For ease of reference, the subscript b on the error term stands for "break" and the subscript f stands for "fixed" (across regimes). By labeling the regressors x_{ft} and x_{bt} as $I(0)$, we mean that the partial sums of the associated noise components satisfy a functional central limit theorem. The conditions imposed are discussed below. We then label a variable as $I(1)$ if it is the accumulation of an $I(0)$ process.

The break points $(T_1, \ldots, T_m)$ are treated as unknown. This is a partial structural change model in which the coefficients of only a subset of the regressors are subject to change while the remaining coefficients are effectively estimated using the entire sample. When $p_f = q_f = 0$, a pure structural change model is obtained where all coefficients are allowed to change across regimes.[1] We can express (1) in matrix form as:

$$Y = G\alpha + \bar{W}\gamma + U$$

where $Y = (y_1, \ldots, y_T)'$, $G = (Z_f, X_f)$, $Z_f = (z_{f1}, \ldots, z_{fT})'$, $X_f = (x_{f1}, \ldots, x_{fT})'$, $U = (u_1, \ldots, u_T)'$, $W = (w_1, \ldots, w_T)'$, $w_t = (z'_{bt}, x'_{bt})'$, $\gamma = (\delta'_{b1}, \beta'_{b1}, \ldots, \delta'_{b,m+1}, \beta'_{b,m+1})'$, $\alpha = (\delta'_f, \beta'_f)'$ and $\bar{W}$ is the matrix which diagonally partitions W at the m-partition $(T_1, \ldots, T_m)$, that is, $\bar{W} = \mathrm{diag}(W_1, \ldots, W_{m+1})$ with $W_i = (w_{T_{i-1}+1}, \ldots, w_{T_i})'$ for $i = 1, \ldots, m+1$. The data generating process is assumed to be

$$Y = G\alpha^0 + \bar{W}^0\gamma^0 + U \tag{2}$$

where α^0, γ^0 and $(T^0_1, \ldots, T^0_m)$ are the true values of the parameters and the matrix $\bar{W}^0$ is the one that partitions W at $(T^0_1, \ldots, T^0_m)$.

As a matter of notation, "$\xrightarrow{p}$" denotes convergence in probability, "$\xrightarrow{d}$" convergence in distribution and "$\Rightarrow$" weak convergence in the space $D[0,1]$ under the Skorohod metric. Also, $x_t = (x'_{ft}, x'_{bt})'$, $u_{xt} = (u^{f\prime}_{xt}, u^{b\prime}_{xt})'$, $z_t = (z'_{ft}, z'_{bt})'$, $u_{zt} = (u^{f\prime}_{zt}, u^{b\prime}_{zt})'$, $\xi_t = (u_t, u^{f\prime}_{zt}, u^{b\prime}_{zt}, u^{f\prime}_{xt}, u^{b\prime}_{xt})'$, $\mu = (\mu'_f, \mu'_b)'$ and $\lambda = \{\lambda_1, \ldots, \lambda_m\}$ is the vector of break fractions defined by $\lambda_i = T_i/T$ for $i = 1, \ldots, m$. We make the following assumptions on the regressors and the elements of the noise component ξ_t.

[1] Note that (1) assumes a particular normalization of the cointegrating vector. Ng and Perron (1997) study the normalization problem in a two variable models. They show that the least squares estimator can have very poor finite sample properties when normalized in one direction but can be well behaved when normalized in the other. This occurs when one of the variables is a weak random walk or is nearly stationary. They suggest to use as regressand the variable for which the spectral density at frequency zero of the first differences is smallest.