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Asymptotic and bootstrap tests for linearity in a TAR-GARCH(1,1) model with a unit root

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ABSTRACT

This paper derives the limiting distribution of the Lagrange Multiplier (LM) test for threshold nonlinearity in a TAR model with GARCH errors when one of the regimes contains a unit root. It is shown that the asymptotic distribution is nonstandard and depends on nuisance parameters that capture the degree of conditional heteroskedasticity and non-Gaussian nature of the process. We propose a bootstrap procedure for approximating the exact finite-sample distribution of the test for linearity and establish its asymptotic validity.

1. Introduction

Threshold autoregressive (TAR) models provide a parsimonious, yet flexible, framework for modeling nonlinearities in time series data. For a survey of the statistical properties of the TAR processes and some recent advances in the inference procedures for TAR models, see Tong (1990) and Hansen (1997, 1999), among others. Chan (1990, 1991) and Chan and Tong (1990) developed a test for linearity and its distribution theory for stationary processes.

The simultaneous presence of high persistence and conditional heteroskedasticity characterizes the dynamic properties of many economic time series (for example, nominal and real interest rates, inflation, real exchange rates, commodity prices etc.). The extension of the test of linearity to nonstationary processes with conditional heteroskedasticity requires new tools for asymptotic analysis. Caner and Hansen (2001) derived the limiting distribution of the Wald test for linearity in unit root but conditionally homoskedastic TAR models. The time-varying conditional variance further complicates the limiting theory of the test statistics. One parsimonious parameterization of the dynamic behavior of the conditional variance is given by the GARCH model (Engle, 1982; Bollerslev, 1986). Wong and Li (1997) obtained the asymptotic distribution of the test for linearity in stationary TAR models with GARCH errors. In another strand of literature, Ling and Li (1998, 2003) and Seo (1999) developed the limiting theory for unit root processes with GARCH disturbances.

This paper builds on the work cited above and derives the limiting distribution of the LM test for linearity in unit root TAR models with GARCH (1,1) errors. We consider two versions of the test based on the information matrix and the robust variance estimator of Bollerslev and Wooldridge (1992). The limiting representations that we obtain depend on nuisance parameters which measure the degree of conditional heteroskedasticity and deviation from normality. To avoid the explicit estimation of these parameters and to improve the small-sample properties of the test, we propose a bootstrap procedure for computing critical (and p-) values and establish its validity.

The main findings about the limiting and the finite-sample behavior of the test for linearity in this setup can be summarized as follows. In the case with normal errors, the limiting distribution appears to be very insensitive to the degree of conditional heteroskedasticity. When the errors are non-normal, the critical values of the non-robust LM test tend to show stronger dependence on the degree of conditional heteroskedasticity while the robust LM test is still almost invariant to changes in the GARCH parameters. The differences in the critical values of the robust LM test for different error distributions are also fairly small. Finally, we find that the robust LM test is numerically better behaved than its non-robust version with superior size properties and higher power for non-normal errors.

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The rest of the paper is organized as follows. Section 2 introduces the model, the assumptions and the LM test statistic for linearity. The limiting distribution of the LM test is derived in Section 3 of the paper. This section also discusses the computation of asymptotic critical values. Section 4 proposes a bootstrap method for approximating the finite-sample distribution of the LM test and establishes its asymptotic validity. Section 5 provides some simulation evidence on the empirical size and power of the tests. In Section 6, we use the proposed procedure to test for possible nonlinearity in the conditional mean of the risk-free interest rate. Section 7 concludes.

2. Model and notation

Consider the TAR(1)-GARCH(1,1) process

$$y_t = \rho y_{t-1} + I_{\{z_{t-1}\geq v\}}(\phi y_{t-1}) + \varepsilon_t \quad (1)$$
$$\varepsilon_t = \sqrt{h_t}\xi_t$$
$$h_t = \omega + \alpha\varepsilon_{t-1}^2 + \beta h_{t-1},$$

where $I_{\{.\}}$ is the indicator function, $v(-\infty < v < \infty)$ is the threshold, z_{t-1} is an observed threshold variable and ϕ is a parameter that measures the difference in the persistence between the two regimes. In model (1), the process y_t is parameterized as switching between two regimes with persistence parameters ρ and $\rho+\phi$ and conditional heteroskedasticity of GARCH(1,1) form that is common to both regimes. Suppose that the following conditions are satisfied.

Assumption 1. Assume that

(a) $\rho = 1$ and $\phi \leq 0$;
(b) z_{t-1} is strictly stationary, ergodic and strong mixing with mixing coefficients α_m satisfying $\sum_{m=1}^{\infty}\alpha_m^{1/2-1/r} < \infty$ for some $r > 2$, with a continuous, strictly increasing, marginal distribution F_z;
(c) $\xi_t \sim iid(0,1)$, $E(\xi_t^3) = 0$, $E(\xi_t^4) = \kappa < \infty$ for all t;
(d) $\Psi = \{(\omega,\alpha,\beta) : 0 < \alpha_l \leq \alpha \leq \alpha_u, 0 < \beta_l \leq \beta \leq \beta_u, \beta^2 + 2\alpha\beta + \kappa\alpha^2 < 1, \omega = 1-\alpha-\beta\}$;
(e) $y_0 = 0$ and h_0 is initialized from its invariant measure.

Part (a) of Assumption 1 parameterizes the conditional mean function of y_t in the two regimes. If $\phi = 0$, the conditional mean dynamics in both regimes is the same and y_t is a linear unit root process. The parameter restriction $\phi \leq 0$ rules out explosive behavior in the second regime. If $\phi < 0$, y_t is a TAR process that switches between a unit root and a mean reverting regime.

The choice of a threshold variable z_{t-1} is crucial for the derivation of the limiting behavior of the test for linearity. In order to facilitate the derivation of the asymptotic results, we follow Caner and Hansen (2001) and impose strict stationarity on the candidate for a threshold variable which would ensure that $\{I_{\{F_Z(z_{t-1})<F_Z(v)\}}\varepsilon_t, \mathcal{F}_{t-1}\}$ forms a stationary and ergodic martingale difference sequence adapted to the σ-field $\mathcal{F}_{t-1}$ generated by $\{(\xi_{t-1}, z_{t-1}), (\xi_{t-2}, z_{t-2}), \ldots\}$. Typically, the threshold variable is chosen to be some function of y_t such as $y_{t-1}-y_{t-2}$ or $|y_{t-1}-y_{t-2}|$. A threshold variable z_{t-d} with $d \geq 1$ can be incorporated in model (1) in a straightforward manner where the delay parameter d often needs to be determined numerically (see Hansen (1997, 1999)). Part (c) of Assumption 1 requires that the standardized errors are symmetric *iid* random variables with finite fourth moment.

The conditions in part (d) ensure that $E(\varepsilon_t^4) < \infty$ and the processes $\{h_t\}$ and $\{\varepsilon_t\}$ are strictly stationary, ergodic and β-mixing with exponential decay (Carrasco and Chen, 2002; Francq and Zakoïan, 2006).[1] In order to simplify the notation in the paper, the last condition in part (d) implies that $E(h_t) = \frac{\omega}{1-\alpha-\beta} = 1$ but this is inconsequential and can be replaced by any constant σ^2. Part (e) specifies the initialization of the conditional mean and variance functions. Assuming y_0 to be fixed at a different value than zero or to be $o_p(T^{1/2})$ does not affect the limiting results derived below. Similarly, the asymptotic distributions are invariant to the assumptions on the initial conditions of ξ and h (Lee and Hansen, 1994; Ling and Li, 2003). The limiting theory in the paper is developed for the simple zero-mean model but we discuss later how to generalize these results to models with deterministic components.[2]

Let $\theta \equiv (\rho,\phi,\delta)' \in \Theta$ be a vector of the unknown parameters, where $\delta = (\omega,\alpha,\beta)$. The Gaussian quasi-likelihood function of this model is given by

$$Q_T(\theta) = \frac{1}{T}\sum_{t=1}^{T} l_t(\theta), \quad (2)$$

where $l_t(\theta) = -\frac{1}{2}\ln h_t - \frac{1}{2}\frac{\varepsilon_t^2}{h_t}$.

To test the null of linearity in the conditional mean, $H_0 : \phi = 0$, we employ the Lagrange Multiplier (LM) test. Let $y_{t-1}(v) = I_{\{z_{t-1}\geq v\}}y_{t-1}$ and $\widetilde{\theta} \equiv (\widetilde{\rho}, 0, \widetilde{\delta})'$ be the constrained MLE[3] in (1) under the null. We consider two forms of the LM test, LM_T and LM_T^R, that are based on two different variance estimators: the information matrix and the robust (sandwich form) estimator of Bollerslev and Wooldridge (1992), respectively. When the value of the threshold is known, the two LM statistics for $H_0 : \phi = 0$ are given by

$$\mathrm{LM}_T = S_T(v)'\left[M_T(v) - M_T(v)M_T^{-1}M_T(v)\right]^{-1}S_T(v) \quad (3)$$

$$\mathrm{LM}_T^R = S_T(v)'\left[R_T(v) - R_T(v)M_T^{-1}M_T(v) - M_T(v)M_T^{-1}R_T(v) + M_T(v)M_T^{-1}R_T M_T^{-1}M_T(v)\right]^{-1}S_T(v), \quad (4)$$

where $S_T(v) = \left.\frac{\partial Q_T(\theta)}{\partial\phi}\right|_{\widetilde{\theta}}$, $R_T = T^{-2}\sum_{t=1}^{T}\left(\left.\frac{\partial l_t(\theta)}{\partial\rho}\right|_{\widetilde{\theta}}\right)^2$, $R_T(v) = T^{-2}\sum_{t=1}^{T}\left(\left.\frac{\partial l_t(\theta)}{\partial\phi}\right|_{\widetilde{\theta}}\right)^2$, $M_T = -T^{-1}\left.\left(\frac{\partial^2 Q_T(\theta)}{\partial\rho^2}\right)\right|_{\widetilde{\theta}}$ and $M_T(v) = -T^{-1}\left.\left(\frac{\partial^2 Q_T(\theta)}{\partial\rho\partial\phi}\right)\right|_{\widetilde{\theta}} = -T^{-1}\left.\left(\frac{\partial^2 Q_T(\theta)}{\partial\phi^2}\right)\right|_{\widetilde{\theta}}$ (for the expressions of these derivatives, see the Appendix).[4]

[1] Some of the intermediate results for our limit theory can be obtained under milder restrictions. For example, Francq and Zakoïan (2006) show the β-mixing of the GARCH(1,1) process without any moment restrictions on ε_t and Ling and Li (2003) derive the limiting distribution of the one-step QMLE of ρ in the linear AR(1)-GARCH(1,1) version of (1) assuming that $E(\varepsilon_t)^2 < \infty$. Our requirement for a finite fourth moment of ε_t is needed in establishing the tightness condition for the GARCH version of the FCLT of a two-parameter process. While this restriction may, in principle, be further relaxed, we do not attempt this in the paper.

[2] In an earlier version of the paper, we reparameterized the coefficient on y_{t-1} as local-to-unity ($\rho_T = 1 + c/T$ for some fixed constant $c \leq 0$) in order to recognize explicitly the uncertainty about the exact value of the AR root that applied researchers typically face. One interesting finding from the local-to-unity parametrization is that the limiting distributions for the test of linearity are very insensitive to local deviations from the unit root. As a result, this version of the paper considers only the case of an exact unit root.

[3] The consistency and the asymptotic normality of the Gaussian quasi-likelihood estimator of the GARCH(1,1) model are established in Lee and Hansen (1994) and Lumsdaine (1996). The consistency and the asymptotic distribution of the parameter ρ are derived in Ling and Li (2003) and Seo (1999).

[4] We also considered a test statistic based on the outer product of the scores estimator of the variance–covariance matrix. The limiting and the numerical properties of this version of the test for linearity are very similar to the robust LM test. We do not present these results here for clarity of exposition and space limitations but they are available from the author upon request.