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Root-n-consistent estimation of weak fractional cointegration

J. Hualde[a], P.M. Robinson[b,*]

[a]School of Economics and Business Administration, Universidad de Navarra, Pamplona 31080, Spain
[b]Department of Economics, London School of Economics, Houghton Street, London WC2A 2AE, UK

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Abstract

Empirical evidence has emerged of the possibility of fractional cointegration such that the gap, β, between the integration order δ of observable time series and the integration order γ of cointegrating errors is less than 0.5. This includes circumstances when observables are stationary or asymptotically stationary with long memory $\left(\text{so } \delta < \frac{1}{2}\right)$ and when they are nonstationary $\left(\text{so } \delta \geqslant \frac{1}{2}\right)$. This "weak cointegration" contrasts strongly with the traditional econometric prescription of unit-root observables and short memory cointegrating errors, where $\beta = 1$. Asymptotic inferential theory also differs from this case and from other members of the class $\beta > \frac{1}{2}$, in particular $\sqrt{n}$-consistent and asymptotically normal estimation of the cointegrating vector v is possible when $\beta < \frac{1}{2}$, as we explore in a simple bivariate model. The estimate depends on γ and δ or, more realistically, on estimates of unknown γ and δ. These latter estimates need to be $\sqrt{n}$-consistent, and the asymptotic distribution of the estimate of v is sensitive to their precise form. We propose estimates of γ and δ that are computationally relatively convenient, relying on only univariate nonlinear optimization. Finite sample performance of the methods is examined by means of Monte Carlo simulations, and several applications to empirical data included.

JEL classification: C32

Keywords: Fractional cointegration; Parametric estimation; Asymptotic normality

*Corresponding author. Tel.: +44 20 7955 7516; fax: +44 20 7955 6592.
E-mail address: p.m.robinson@lse.ac.uk (P.M. Robinson).

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1. Introduction

Cointegration analysis has usually proceeded under the assumption of unit-root ($I(1)$) observable series and short-memory stationary ($I(0)$) cointegrating errors. Here, the least squares estimate (LSE) of the cointegrating vector is not only consistent, but super-consistent (with convergence rate equal to sample size, n) (Stock, 1987) despite contemporaneous correlation between regressors and cointegrating errors; optimal estimates, which account for this correlation, enjoy no better rate of convergence (Phillips, 1991).

It is also possible to consider cointegration in a fractional context. To be specific, we consider the model

$$\left.\begin{array}{l} \Delta^{\gamma}(y_t - \nu x_t) = u_{1t}^{\#}, \quad t \geqslant 1, \;\; y_t = 0, \;\; t \leqslant 0, \\ \Delta^{\delta} x_t = u_{2t}^{\#}, \quad t \geqslant 1, \;\; x_t = 0, \;\; t \leqslant 0 \end{array}\right\} \tag{1}$$

for the bivariate observable sequence $\{y_t, x_t\}$. Here $\Delta = 1 - L$, where L is the lag operator;

$$(1-L)^{-\alpha} = \sum_{j=0}^{\infty} a_j(\alpha) L^j, \quad a_j(\alpha) = \frac{\Gamma(j+\alpha)}{\Gamma(\alpha)\Gamma(j+1)}, \tag{2}$$

taking $\Gamma(\alpha) = \infty$ for $\alpha = 0, -1, -2, \ldots$, and $\Gamma(0)/\Gamma(0) = 1$; the # superscript attached to a scalar or vector sequence v_t has the meaning

$$v_t^{\#} = v_t 1(t > 0), \tag{3}$$

where $1(\cdot)$ is the indicator function; $\{(u_{1t}, u_{2t}), t = 0, \pm 1, \ldots\}$ is an unobservable covariance stationary bivariate sequence having zero mean and spectral density matrix that is nonsingular and bounded at all frequencies; and the real numbers γ and δ satisfy

$$0 \leqslant \gamma < \delta. \tag{4}$$

On this basis, we refer to $u_t = (u_{1t}, u_{2t})'$ as $I(0)$, x_t as $I(\delta)$ and $y_t - \nu x_t$ as $I(\gamma)$, while for

$$\nu \neq 0 \tag{5}$$

(4) implies that y_t is also $I(\delta)$; under (1), (4) and (5), y_t and x_t are said to be cointegrated $CI(\delta, \gamma)$ (Engle and Granger, 1987), for which it is necessary that y_t and x_t share the same integration order (the argument of $I(\cdot)$). The truncations on the right-hand side in (1) ensure that the model is well-defined in the mean square sense, whereas, for example, $\Delta^{-\delta} u_{2t}$ does not have finite variance when $\delta \geqslant \frac{1}{2}$.

We anticipate

$$Cov(u_{1t}, u_{2t}) \neq 0, \tag{6}$$

when, rewriting the first equation of (1) as the regression

$$y_t = \nu x_t + v_{1t}, \quad v_{1t} = \Delta^{-\gamma} u_{1t}^{\#}, \tag{7}$$

the x_t and v_{1t} are contemporaneously correlated. When

$$\delta < \tfrac{1}{2} \tag{8}$$

(6) leads to inconsistency of the LSE due to the fact that x_t is asymptotically stationary and so its sum of squares does not asymptotically dominate that of v_{1t}. To overcome this problem, Robinson (1994a) showed that a narrow-band frequency-domain least squares