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# Free vibration of a rotating tapered Rayleigh beam: A dynamic stiffness method of solution

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ABSTRACT

The dynamic stiffness method for free vibration analysis of a rotating tapered Rayleigh beam is developed to investigate its free vibration characteristics. The type of taper considered covers a majority of practical cross-sections. The effects of centrifugal stiffening, an outboard force, an arbitrary hub radius and importantly, the rotatory inertia (Rayleigh beam) are included in the analysis. Natural frequencies and mode shapes of some examples are illustrated by using the developed dynamic stiffness matrix and applying the Wittrick–Williams algorithm. The theory is validated by using comparative results in the literature. The effects of slenderness ratio, rotational speed and taper ratio on results are discussed. This is followed by some concluding remarks.



## 1. Introduction

There are numerous engineering applications of rotating beams. Typical examples include helicopter, compressor and turbine blades for which the computation of natural frequencies and mode shapes plays an important role in their design. The cross-sections of these structures in real life are quite complex, but for simplicity and also to establish trends and to make some engineering judgements, researchers have often treated them in a relatively simple manner. For instance, the coupling between various modes of elastic deformations has often been ignored. In particular, the free vibration behaviour of rotating uniform or tapered beams undergoing only bending deformation (which is generally the most predominant component in helicopter or wind turbine blades) has been considered by a substantial number of research workers. Research based on such assumption that the beam deforms only in bending is no-doubt an oversimplification, but nevertheless, there is some justification for doing this research, particularly when predicting the general behaviour of the structure and also to gain some insights into the problem. The literature on the free vibration behaviour of rotating beams using uncoupled bending theory is quite voluminous and surprisingly, is still growing. The authors have compiled a selective sample of recent publications [1–19] in chronological order which is appended to this paper so that interested readers can obtain background information and access useful cross-references on the subject.

The development of rotating beam theories is understandably preceded by the development of non-rotating beam theories and predictably there are some interesting parallels. The Bernoulli–Euler beam theory of the eighteenth century [20], the Rayleigh beam theory of the nineteenth century [21] and the Timoshenko beam theory of the early twentieth century [22,23] have no-doubt played very important roles in paving the way for the corresponding research into the free vibration behaviour of rotating beams [1–19]. From a historical perspective, it is probably true that Timoshenko's beam theory [22,23] which accounts for the effects of both shear deformation and rotatory inertia of beam cross-section has in many ways, overshadowed the relatively unknown Rayleigh beam theory which was developed by Lord Rayleigh [21] many years earlier. The reason for this is obvious because the latter accounts for the effect of the rotatory inertia only, whereas the former includes both the effects of the rotatory inertia as well as the shear deformation of the beam cross-section. Furthermore, the Rayleigh beam theory [21] for non-rotating beam has often been overlooked because of its relative simplicity and ease of application when compared to Timoshenko's beam theory. However, for rotating tapered beams, the development of the Rayleigh beam theory is not so obvious and as it will be shown later the investigation is quite complex leading to an important development. This development cannot be ignored or sidestepped prior to the dynamic stiffness development of rotating tapered Timoshenko beams which no-doubt will be a tremendously difficult task. The difficulty arises because the governing differential equations for flexural displacement and bending rotation are quite different and very complicated for a tapered Timoshenko beam and

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furthermore, to add to the complexity, they are coupled unlike the one in the Rayleigh beam theory which principally focuses on just only one differential equation related to the bending displacement alone. It appears that the difficult task of developing the dynamic stiffness theory for a rotating tapered Timoshenko beam can be significantly facilitated by the corresponding development of a rotating tapered Rayleigh beam. Thus the purpose of this paper is to undertake the task of developing the dynamic stiffness matrix of a rotating tapered Rayleigh beam and then to carry out its free vibration analysis.

The literature on the free vibration of rotating uniform or tapered beams shows that a majority of the publications rely on classical methods based on the solution of the governing differential equations and the subsequent substitution of boundary conditions which ultimately leads to the frequency equation. However, some finite element method based solutions [7,12,16] are also available amongst the application of other methods such as the differential transform method [10,17,18]. A significant contribution to the literature in recent years is indeed the application of the dynamic stiffness method (DSM) [2,5,11] which extends the free vibration analysis of rotating beams to a much wider context. This is because the DSM has all the important features of the finite element method (FEM), but importantly, unlike the FEM, it allows an exact free vibration analysis of structures possible. Thus the DSM has been successfully applied to rotating uniform Bernoulli–Euler and Timoshenko beams [2,5]. These developments have later been extended to cover rotating tapered Bernoulli–Euler beams [11]. The DSM development for a rotating tapered Rayleigh beam is therefore, a natural extension of previous research. This useful extension is far from being trivial as will be shown later, requiring considerable theoretical and computational efforts.

Within the above pretext, the dynamic stiffness matrix of a rotating tapered Rayleigh beam is developed and a free vibration analysis is carried out. The range of problems considered includes beams with linearly varying taper in depth and/or width of the cross-section. In terms of cross-sectional properties this essentially means that the area and the second moment of area of the beam can vary in two different ways. In the first case when either the depth or the width (but not both) of the beam varies linearly along the length, the corresponding variation of the area of cross-section will be linear whereas the variation of the second moment of area will be cubic over the length. On the other hand, for the second case when both the depth and the width vary linearly, the variation of the cross-sectional area will follow a square law whereas the second moment of area variation will be of fourth power. Using these two types of property variations, a large number of cross-sections can be constructed [24,25] which cover a huge number of practical cases. For instance, a linearly varying tapered beam with thin-walled circular cross-section of constant thickness falls into the former category whereas the one with a solid circular cross-section will belong to the latter.

The investigation proceeds as follows. First the governing differential equation of motion in free vibration of a rotating tapered Rayleigh beam with linearly varying taper is derived using Hamilton's principle. For harmonic oscillation, the differential equation is solved using Frobenius method of series solution [26]. The expressions for bending displacement, bending rotation, shear force and bending moment are formulated in explicit algebraic form. Next, the boundary conditions are applied to relate the amplitudes of the nodal forces of the freely vibrating rotating tapered Rayleigh beam to those of the corresponding displacements via the frequency dependent dynamic stiffness matrix relationship. Finally the well-established algorithm of Wittrick and Williams [27] is applied to the resulting dynamic stiffness matrix to compute the natural frequencies and mode shapes of some illustrative examples.

## 2. Theory

In a right-handed Cartesian co-ordinate system, Figs. 1 and 2 show respectively, the two types of taper considered in this paper for a rotating tapered Rayleigh beam. The $Y$-axis coincides with the axis of the beam as shown whereas the $Z$-axis is parallel but not necessarily coincidental with the axis of rotation. It is assumed that the beam is rotating at a constant angular velocity $\Omega$ with an arbitrary hub radius $r_H$ as shown. Fig. 1 shows a linear variation of depth and a constant width of the cross-section along the length whereas Fig. 2 shows a linear variation of both width and depth. Thus the variations of cross-sectional area and second moment of area are linear and cubic for the former (Fig. 1) whereas those for the latter are of second and fourth order variations (Fig. 2), respectively.

If $L$ is the length, $c$ is the taper ratio, $A(y)$ and $I(y)$ are the area and second moment of area of the cross-section at a distance $y$, then the variations of the area $A(y)$, and the second moment of area $I(y)$ for both types of tapered beam can be expressed by using the following formulas.

$$A(y) = A_0\left(1 - c\frac{y}{L}\right)^n \tag{1}$$

$$I(y) = I_0\left(1 - c\frac{y}{L}\right)^{n+2} \tag{2}$$

where $A_0$ and $I_0$ are the area and the second moment of area at the left-hand end, i.e. the thick-end of the beam, respectively. The integer $n$ takes the value 1 for the first type (see Fig. 1) and 2 for the second type (see Fig. 2) of taper described by Eqs. (1) and (2). A large number of cross-sections can be constructed by using these two values of $n$ (see Refs. [24,25]), covering many practical cases. However, the rectangular cross-section is shown in Figs. 1 and 2 only for convenience.

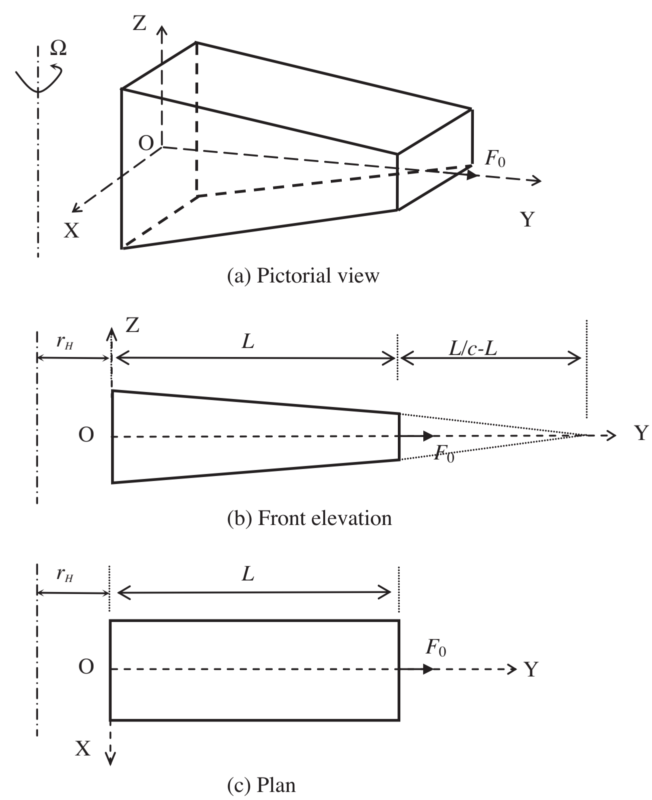


**Fig. 1.** A rotating tapered beam with a constant width and linearly varying depth.