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Journal of Econometrics 133 (2006) 863–886
JOURNAL OF Econometrics
www.elsevier.com/locate/jeconom

A consistent bootstrap test for conditional density functions with time-series data

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Available online 15 August 2005

Abstract

This paper presents a new test for evaluating *conditional* density functions for time-series data, thereby being applicable to forecasting problems. We show that the test statistic is asymptotically distributed standard normal under the null hypothesis, and diverges to infinity when the null hypothesis is false. We use a bootstrap algorithm to approximate the distribution of the test statistic, and show that the bootstrap distribution converges to the asymptotic distribution of the test statistic in probability. An application to inflation forecasting is also presented to demonstrate the usefulness of the test.

JEL classification: C12; C15; E37

Keywords: Bootstrap; Conditional density function; Density forecasting

1. Introduction

Conditional density functions arise in a variety of areas. One of the more useful applications involves density forecasting, where the probability density of the forecast of a time series, such as the rate of inflation, can be used to make probability

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0304-4076/$ - see front matter
doi:10.1016/j.jeconom.2005.06.016

statements regarding the future course of that series.[1] However, the probability density, and its resulting interpretation, is conditional on the hypothesis that the model used to produce the forecasts is correctly specified. A test is thus required to determine whether the *conditional* density function implied by the model corresponds to the one implied by the data.

Tests for evaluating conditional densities have a recent history, beginning with Andrews (1997) who proposes a conditional Kolmogorov test. The asymptotic null distribution of this test is dependent upon a nuisance parameter, so the critical values are obtained by a bootstrap procedure. Meanwhile, Zheng (2000) uses the Kullback–Leibler information criterion as a basis for testing conditional density functions. Zheng's test is consistent against all alternatives to the null, but the simulation results revealed that the power and size are sensitive to the smoothing parameters. A limitation of the above tests is that the data must be independently and identically distributed, thereby ruling-out time-series applications.

In the area of time-series, Diebold et al. (1998) propose a method for evaluating conditional density forecasts based on an integral transform of the conditional density function. To evaluate a density forecast, they examine whether the transformed series resembles a i.i.d. uniform distribution on the [0,1] interval. If it does, then the null hypothesis of a correctly specified conditional density function is accepted. However, this approach requires a visual assessment of histograms, and is limited by the requirement that there be no unknown parameters under the null hypothesis. In other words, the absence of a test statistic prevents the implementation of formal statistical inference.

Recently, some authors have attempted to develop formal statistical tests of conditional density functions in the diffusion process literature. Thompson (2002) and Hong and Li (2002) apply the Diebold et al. (1998) methodology of transforming the observed data via a probability integral transform. Thompson's (2002) test is based on the empirical distribution function, which is computationally convenient. To obtain the critical values, Thompson (2002) uses some upper bounds. Use of upper bound critical values may be too conservative, adversely affecting the power of the test. Hong and Li (2002) propose two nonparametric transition density-based specification tests by comparing a kernel estimator for the joint density of the transformed data with unity. Their tests can be applied to both time-inhomogeneous and nonstationary diffusion processes. When no prior information about serial dependence of lag order is available, they have to use a family of test statistics with different lags to determine at which lags the independent and identical uniform property is violated.

In this paper we propose a consistent computationally convenient test for evaluating conditional density functions for time series data. The test statistic is based on the integrated squared difference between the conditional density function implied by the parametric model and a nonparametric estimator of the true

[1] In its *Inflation Report*, the Bank of England regularly displays density forecasts for inflation over the next 2 years through the use of fan charts. For details on the construction of fan charts, see p. 52 of the February 1999 *Inflation Report*.