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A model reduction technique for beam analysis with the asymptotic expansion method

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ABSTRACT

In this paper, we apply the asymptotic expansion method to the mechanical problem of beam equilibrium, aiming to derive a new beam model. The asymptotic procedure will lead to a series of mechanical problems at different order, solved successively. For each order, new transverse (in-plane) deformation and warping (out of plane) deformation modes are determined, in function of the applied loads and the limits conditions of the problem. The presented method can be seen as a more simple and efficient alternative to beam model reduction techniques such as POD or PGD methods. At the end of the asymptotic expansion procedure, an enriched kinematic describing the displacement of the beam is obtained, and will be used for the formulation of an exact beam element by solving analytically the arising new equilibrium equations. A surprising result of this work, is that even for concentrated forces (Dirac delta function), we obtain a very good representation of the beam's deformation with only few additional degrees of freedom.

1. Introduction

The development of higher order beam model that are able to represent accurately the transversal deformation and the warping of the cross section, is of great interest for bridge and structural engineers. These kind of elements have the advantage to contain in a single beam model, the global and local response of the structure, instead of multiple models with different types of elements. Thus, using higher order beam elements will result in a major time reduction and work simplification for their users. In practice however, there are very few engineering offices or software editors for bridge analysis that uses these kind of elements, mainly because the many necessary aspects for bridge analysis, such as shrinkage, creep, and skewed curved slab, are not yet completely solved for these elements, and still the subject of an intense research work.

Many higher order beam element has been developed using different approach [9–17], for example the GBT (generalized beam theory) theory, developed mainly for thin walled profiles, the VABS (variational asymptotic beam sectional analysis) method, all giving very good results in comparison with reference models made with shell or 3D finite elements. In all the aforementioned methods, the beam model is derived in two step as in [1,2]. First, a cross section analysis is performed to determine an appropriate kinematics (or a basis in which the model is reduced) and then the equilibrium equations are solved, generally by using one of the available numerical method (FEM, BEM, etc.).

In a previous work [1,2], a new beam model was presented and its derivation was performed in two independent steps. The first one is the construction of an enriched kinematic of the beam, by using transverse deformation modes obtained from an eigenvalue analysis of the cross section, and their associated warping modes, obtained from an iterative equilibrium schemes. Once the kinematic is determined, the second step consists in using the principle of virtual work to obtain the new equilibrium equations associated to the newly introduced degrees of freedom. These equations were solved analytically in order to assemble the stiffness matrix of the element. The advantages of the beam model in [1], in comparison to other higher order beam elements present in the literature, is its validity for thick and thin-walled profile and its stiffness matrix derived from an analytic solution of the equilibrium equations, which means that there is no need of meshing the beam model. Nevertheless, the main weakness of the element is that the transverse deformation and warping modes representing the kinematic, are not specific to the applied loadings, and thus may or may not have an effect on the beam's response, which may results only in adding unnecessary degrees of freedom to the system. From here originate the idea to determine specific modes for the external

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loading that will have by construction an effect on the beam's response. These modes will allow us to obtain the same accurate results in [1], but with much less additional degrees of freedom.

The asymptotic expansion method (AEM) [3–8,19–21] is one of the many available mathematical tools to solve general differential equations, and thus can be used in all fields where these kind of equations appears. In structural mechanics, the AEM have been employed to find new solutions for beam or shell structures, or to find a rigorous justification for some existing models [20,21]. An implicit assumption, used in all beam theories, which is a separation of the longitudinal variation with the transversal one in the description of the kinematic, found a justification from the AEM by assuming a variable separation for the applied external forces. This assumption is the key element in this work, since unlike the classical work of Trabucho and Viano [19], it will be exploited for the higher order modes, to obtain from the general 3D equilibrium problem of the beam, a series of problems expressed only on the 2D cross section. The AEM can then be seen as a more simple and efficient alternative to other model reduction techniques such as POD (Proper Orthogonal Decomposition) and PGD (Proper Generalized Decomposition), for which a complete 3D pre-analysis of the beam needs to be performed, whereas only a cross section pre-analysis is necessary for the AEM to determine the prominent modes into the beam's response, under a definite loading.

In Section 2 of this work, we give a summary of the previous work developed in [1], and then in Section 3 the objective of the paper is stated. In Section 4, the AEM is applied to a beam element with a constant cross-section, and at the end of the procedure an enriched kinematic of the beam in function of external loads is determined. Finally in Section 5, the principle of virtual work is used to derive the new equilibrium equations, which will be solved analytically, as in [1], to obtain the stiffness matrix of the beam. The numerical results obtained from the developed beam model will be compared to reference models with shell or brick elements. The AEM has proven to be a very efficient method to determine the minimum basis (or kinematic) representing the deformation of the beam.

1.1. Notations

We set first the notations used in this paper:

- The letters written in bold will designates tensors or vectors.
- The coordinates system attached to the beam will be expressed by (x_1, x_2, x_3) or (y_1, y_2, y_3), where the longitudinal component is represented by x_3 or y_3 and the transversal component by (x_1, x_2) or (y_1, y_2). See figure below (Fig. 1).
- The repeated index summation convention will always be used (unless the contrary is explicitly specified), where the latin letters indices can take the values from 1 to 3 and the greek letters indices the values 1 and 2:

$$a_i b_i = \sum_{i=1}^{3} a_i b_i, \quad a_\alpha b_\alpha = \sum_{\alpha=1}^{2} a_\alpha b_\alpha$$

- For the derivation we use the following notations:

$$a_{,1} = \partial_1 a = \frac{\partial a}{\partial y_1}, \quad a_{,2} = \partial_2 a = \frac{\partial a}{\partial y_2}, \quad a_{,\alpha} = \partial_\alpha a = \frac{\partial a}{\partial y_\alpha}$$

$$a_{,ij} = \partial_i \partial_j a = \frac{\partial^2 a}{\partial y_i \partial y_j}, \quad a^{(n)} = \frac{\partial^n a}{\partial y_3^n}$$

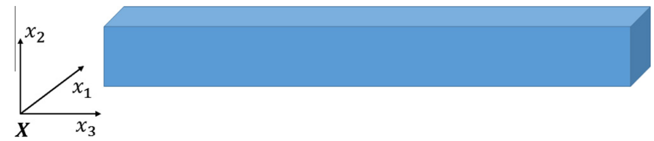

Fig. 1. A frame attached to a rectangular beam.

2. Summary of previous work

Before developing the asymptotic expansion method for a beam element, we explain in this section the general philosophy of the paper. Our goal is to develop a beam element capable of representing accurately the transverse deformation and the warping of the cross sections under arbitrary loadings, and thus to obtain with the beam model, equivalent results to those of a shell or brick models. The starting point for the development of a beam element is not the equilibrium equations, but its kinematic, which is the key element of every beam model, especially in elasticity. For example in the well-known Euler–Bernoulli beam theory, the model fails to give the shear strain because of the choice made on the kinematic, which is that the rotation of the cross section is equal to the deflection derivate. Another example concerns the torsion in both Timoshenko and Euler–Bernoulli beam theory. In those two models the torsion inertia is equal to the polar inertia, which is wrong in most cases. Thus, the precision of the model will depends crucially on the adopted kinematic.

In [1], a kinematic was proposed to describe the arbitrary transverse deformation and warping of the cross section. To determine the transverse deformation modes, the beam cross section is discretized with 2D triangular elements for which an elementary stiffness matrix is associated. Assembling all of these elementary matrices, the global stiffness matrix $\boldsymbol{K}_s$ of the section is obtained, for which a standard eigenvalue analysis is performed:

$$\text{find } (\lambda, \boldsymbol{v})/\boldsymbol{K}_s \boldsymbol{v} = \lambda \boldsymbol{v} \tag{1}$$

The eigenvectors will form a basis of transverse deformation modes that will be used to enrich the kinematic, and where an arbitrary transverse deformation will be decomposed. We note that by construction of this basis, the modes are linearly independent, and that the strain energy U associated to a mode is given by:

$$U = \frac{1}{2}\boldsymbol{v}^T \boldsymbol{K}_s \boldsymbol{v} = \frac{1}{2}\lambda \boldsymbol{v}^T \boldsymbol{v} = \frac{1}{2}\lambda \tag{2}$$

From Eq. (2), it can be deduced that the modes with the lowest eigenvalues mobilizes less energy, and thus have more chances to occur. From this argument, if we want to determine a basis of n transverse deformation functions to enrich our kinematic, we will use then the n eigenvectors with the lowest eigenvalues. But the main drawback of this procedure, is that the selected modes may not participate to the beam's response, depending on the applied loads and boundary conditions.

For the determination of the warping mode basis in [1], we start by deriving for each transverse deformation mode, an associated warping mode of the first order, by making the assumption of uniform warping. To develop the procedure, let us consider the following kinematic:

$$\boldsymbol{d} := \begin{Bmatrix} \psi_1 \zeta \\ \psi_2 \zeta \\ u \end{Bmatrix} \tag{3}$$

where $\boldsymbol{\psi} = (\psi_1(\boldsymbol{x}), \psi_2(\boldsymbol{x}))$ is a known arbitrary transverse deformation mode, $\zeta(x_3)$ its associated degree of freedom, and $u(\boldsymbol{X})$ the longitudinal displacement that has an unknown form for the moment and needs to be defined.