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The method of fundamental solutions for an inverse boundary value problem in static thermo-elasticity

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ABSTRACT

The inverse problem of coupled static thermo-elasticity in which one has to determine the thermo-elastic stress state in a body from displacements and temperature given on a subset of the boundary is considered. A regularized method of fundamental solutions is employed in order to find a stable numerical solution to this ill-posed, but linear coupled inverse problem. The choice of the regularization parameter is based on the L-curve criterion. Numerical results are presented and discussed.

1. Introduction

Whenever a solid is subject to heating conditions that give rise to a temperature distribution which produces thermal expansions throughout its volume, the structure is subject to thermo-elastic loadings. In much experimental research concerning the determination of thermo-elastic fields in a nuclear reactor or in structures of spacecraft and propulsion systems, measurements are possible on only an accessible part of the surface of the body. The remaining surface is usually in contact with a hostile environment and it is therefore very difficult or even impossible to place thermocouples, heat flux probes, or strain gauges on it. In such a situation, one has to find the thermo-elastic stress state in the body by using displacements and temperature measurements taken on a subset of the boundary. In this study, this inverse problem of coupled thermo-elasticity in the static regime is solved numerically, apparently, for the first time. For related Cauchy inverse boundary condition numerical reconstructions in static thermo-elasticity the reader is referred to [3–5].

Since the problem is linear and we assume that the thermal and mechanical properties of the material are constant, the numerical technique we choose to employ to solve this problem is the method of fundamental solutions (MFS) which offers several advantages over the more established boundary element method (BEM) [24]. In particular, as stated in [11], the MFS is meshless in the sense that only a collection of points is required for the discretization of the problem under investigation. Unlike the BEM, no potentially troublesome integration is required in the MFS. These features make the MFS very easy to implement, in particular for problems in complex geometries and three dimensions. Moreover, unlike domain discretization methods such as the finite element (FEM) or finite difference (FDM) methods, it is a boundary method which means that only the boundary of the solution domain needs to be considered. This makes it particularly attractive for the solution of boundary value problems in which the boundary is of prime interest, such as inverse problems and free boundary problems. Finally, like the BEM it can easily deal with infinite domains by incorporating the behaviour of the solution of the problem at infinity into the fundamental solution of the governing equation. Because of its advantages, the MFS has been used extensively over the last decade for the solution of inverse problems [11]. In the particular problem under investigation because of the way the particular solution is derived it appears natural to use the MFS. The disadvantage of the MFS is that, like the BEM, it cannot be readily applied to problems in which the fundamental solution of the operator in the governing equation is not known explicitly or to inhomogeneous equations. In addition, a disadvantage of the MFS over the BEM is that the optimal location of the pseudo-boundary on which the singularities are to be placed is, in general, not known. We finally mention that a few different meshless boundary discretization techniques, different than, but related to the MFS have recently been introduced by Chen and his co-workers, see e.g., [6,7].

The mathematical formulation of the problem is given in Section 2. In Section 3 we describe the MFS for the solution of

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the problem in question. Furthermore, the Tikhonov regularization method with the choice of the regularization parameter given by the L-curve criterion is employed in order to obtain a stable solution. This stability is investigated in Section 4 with respect to noise in the input data. Finally, the conclusions drawn from this work are presented in Section 5.

2. Mathematical formulation

Consider a linear-elastic homogeneous, mechanically and thermally isotropic body occupying a simply connected domain Ω bounded by a smooth boundary $\partial\Omega$. The body is subjected to an unknown temperature field with sources outside Ω and we assume that other internal heat sources or body forces are absent. We consider practical applications involving high temperatures or hostile environments in which a part of the boundary $\Gamma_2 \subset \partial\Omega$ is inaccessible to measurements. In this case measurements of the displacements and temperature are available on the remaining accessible part $\Gamma_1 = \partial\Omega \setminus \Gamma_2$. Then the inverse problem of coupled static thermo-elasticity requires finding the displacement $\boldsymbol{u}$ and the temperature T satisfying the Navier–Lamé system [16,20]

$$\mathcal{L}\boldsymbol{u} + \overline{\gamma}\nabla T = 0, \quad \text{in } \Omega \tag{1}$$

and the steady-state heat conduction Laplace equation

$$\kappa \Delta T = 0, \quad \text{in } \Omega, \tag{2}$$

where $\kappa > 0$ is the thermal conductivity, $\overline{\gamma} = \frac{2G\overline{\alpha}_T(1+\overline{\nu})}{1-2\overline{\nu}}$, G is the shear modulus,

$$\overline{\nu} = \begin{cases} \nu & \text{plane strain} \\ \nu/(1+\nu) & \text{plane stress} \end{cases} \quad \text{and} \quad \overline{\alpha}_T = \begin{cases} \alpha_T & \text{plane strain,} \\ \alpha_T(1+\nu)/(1+2\nu) & \text{plane stress,} \end{cases} \tag{3}$$

where α_T is the coefficient of thermal expansion, ν is the Poisson ratio and

$$\mathcal{L}\boldsymbol{u} = -G\left[\nabla\cdot\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\mathsf{T}}\right) + \frac{2\overline{\nu}}{1-2\overline{\nu}}\nabla(\nabla\cdot\boldsymbol{u})\right]. \tag{4}$$

In (4) and in the sequel the superscript T denotes the transpose of the matrix $\nabla\boldsymbol{u}$. The governing equations (1) and (2) have to be solved subject to the coupled boundary conditions

$$\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} - \overline{\gamma}T\boldsymbol{n} = \boldsymbol{f}, \quad \text{on } \partial\Omega, \tag{5}$$

where $\boldsymbol{n}$ is the outward unit normal to $\partial\Omega$ and the stress tensor is

$$\boldsymbol{\sigma}(\boldsymbol{u}) = 2G\left[\boldsymbol{\varepsilon} + \frac{\overline{\nu}}{1-2\overline{\nu}}\text{tr}(\boldsymbol{\varepsilon})I\right] \tag{6}$$

and

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\mathsf{T}}\right) \tag{7}$$

is the strain tensor. In (6), $\text{tr}(\boldsymbol{\varepsilon})$ denotes the trace of the strain tensor and I is the identity matrix. The left-hand side of Eq. (5) represents the traction vector $\boldsymbol{t}$, whilst its right-hand side $\boldsymbol{f}$ is a known vector function.

If T is prescribed on the whole boundary $\partial\Omega$ then, this gives rise to the direct problem in thermo-elasticity which is well-posed up to a rigid body displacement. However, in our inverse problem only the part Γ_1 of the boundary $\partial\Omega$ is accessible to measurement and on it we prescribe both the temperature and the displacement, namely,

$$T = \widetilde{T}, \quad \text{on } \Gamma_1, \tag{8a}$$

$$\boldsymbol{u} = \tilde{\boldsymbol{u}}, \quad \text{on } \Gamma_1. \tag{8b}$$

The uniqueness of solution of the inverse problem (1), (2), (5) and (8) was proved in [13,14]. However, the problem is still ill-posed since small errors in the input data (8a) and (8b) cause large errors in the output solution $(\boldsymbol{u}, T)$ in Ω and, especially in the heat flux

$$q := -\kappa\nabla T\cdot\boldsymbol{n} \quad \text{on } \Gamma_2 \tag{9a}$$

and traction

$$\boldsymbol{t} := \boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} - \overline{\gamma}T\boldsymbol{n}, \quad \text{on } \Gamma_2. \tag{9b}$$

It is worth mentioning, that a related but decoupled Cauchy inverse and ill-posed problem consisting of Eqs. (1) and (2) with (8a) and (8b) and

$$-\kappa\nabla T\cdot\boldsymbol{n} = \tilde{q} \quad \text{on } \Gamma_1, \tag{10a}$$

$$\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} - \overline{\gamma}T\boldsymbol{n} = \tilde{\boldsymbol{t}}, \quad \text{on } \Gamma_1, \tag{10b}$$

see [3], has recently been solved using the MFS combined with the Tikhonov regularization in [18]. We also remark that from (5) and (8a) we obtain

$$\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} = \boldsymbol{f} + \overline{\gamma}\widetilde{T}\boldsymbol{n}, \quad \text{on } \Gamma_1, \tag{11}$$

which together with (8b) form a set of Cauchy data prescribed on Γ_1.

3. The method of fundamental solutions (MFS)

The combination of the MFS with the method of particular solutions (MPS) for approximating a pair solution $(\boldsymbol{u}, T)$ of (1) and (2) is described in [17,12], in two and three dimensions, respectively.

In this section, we shall consider the two-dimensional case and describe the regularized MFS for solving the inverse problem (1), (2), (5) and (8).

First, in the MFS for the Laplace equation (2) we seek the approximation of the temperature as

$$T_N(\boldsymbol{x}) = \sum_{\ell=1}^{N} c_\ell F(\boldsymbol{x}, \xi_\ell), \quad \boldsymbol{x}\in\overline{\Omega}, \tag{12}$$

where the sources $\xi_\ell \in \mathbb{R}^2 \setminus \overline{\Omega}$ and

$$F(\boldsymbol{x}, \xi_\ell) = -\frac{1}{2\pi\kappa}\log|\boldsymbol{x} - \xi_\ell| \tag{13}$$

is the fundamental solution of the two-dimensional Laplace equation (2).

Introducing (12) into (1) yields

$$\mathcal{L}\boldsymbol{u}(\boldsymbol{x}) = \frac{\overline{\gamma}}{2\pi\kappa}\sum_{\ell=1}^{N} c_\ell \frac{(x_1 - \xi_{1,\ell}, x_2 - \xi_{2,\ell})}{|\boldsymbol{x} - \xi_\ell|^2}, \quad \boldsymbol{x} = (x_1, x_2)\in\Omega, \tag{14}$$

where $\xi_\ell = (\xi_{1,\ell}, \xi_{2,\ell})$ for $\ell = \overline{1, N}$. Then, using the MPS one seeks, see [17],

$$\boldsymbol{u} = \boldsymbol{u}_H + \boldsymbol{u}_P, \tag{15}$$

where

$$\mathcal{L}\boldsymbol{u}_H = \boldsymbol{0} \quad \text{in } \Omega, \tag{16}$$

and

$$\boldsymbol{u}_P(\boldsymbol{x}) = -\frac{\overline{\alpha}_T}{4\pi\kappa}\left(\frac{1+\overline{\nu}}{1-\overline{\nu}}\right)\sum_{\ell=1}^{N} c_\ell(\boldsymbol{x} - \xi_\ell)\log|\boldsymbol{x} - \xi_\ell|, \quad \boldsymbol{x}\in\Omega \tag{17}$$

is a particular solution of (14).

We further apply the MFS to the Lamé homogeneous system (16) to approximate, see [15],

$$\boldsymbol{u}_H(\boldsymbol{x}) = \sum_{\ell=1}^{N} U(\boldsymbol{x}, \xi_\ell)\boldsymbol{d}_\ell, \quad \boldsymbol{x}\in\overline{\Omega}, \tag{18}$$

where $\boldsymbol{d}_\ell = (d_{1_\ell}, d_{2_\ell})^{\mathsf{T}}$ and