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Improved singular boundary method for elasticity problems

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ABSTRACT

This paper presents a simple approach to evaluate origin intensity factors of the singular boundary method (SBM), a recent strong-form boundary discretization numerical technique. The SBM overcomes the perplexing 'fictitious boundary issue' associated with the method of fundamental solutions (MFS) and in it the source points and collocation points coincide on the real physical boundary. By analogy with the boundary element method (BEM), we develop a desingularization strategy for the direct computation of singular kernels in the SBM, without losing the merits of being truly meshless, integration-free, and easy-to-implement. In addition, an efficient non-linear co-ordinate transformation is employed to tackle the near singularities of the kernel functions, when the calculation point is close to, but not on, the boundary. It is shown that the proposed SBM fully inherits the merits of the BEM and MFS. The advantages, disadvantages and potential applications of the proposed method, as compared with the MFS and the BEM, are also examined and discussed in detail.

1. Introduction

The method of fundamental solutions (MFS) belongs to the family of meshless boundary collocation methods that present remarkable results with a small computational effort [1–3]. The MFS is easy-to-implement and computationally efficient and thus a competitive alternative for the solution of boundary value problems. However, the traditional MFS requires a fictitious boundary outside the problem domain in order to avoid singularities of the fundamental solutions. Despite many years of research, the determination of the fictitious boundary is largely based on experience and presents the most serious drawback in the MFS applications to real-world problems [4–6].

In recent decades, considerable efforts have been made to mitigate this difficulty associated with the MFS, so that the source points can directly be placed on the real boundary. Chen and Tanaka [7] employed a nonsingular general solution instead of the singular fundamental solutions for two-dimensional (2D) Laplace and Helmholtz problems. Yong and his co-workers [8] applied a desingularization method of subtracting and adding-back technique for 2D potential problems. Sarler [9] proposed a similar method to determine the diagonal coefficients of the weakly and strong singular kernels by the integration of the fundamental solution on line segments and the inverse interpolation technique, respectively. Liu [10] applied successfully a desingularization strategy, based on the boundary element formulation, for 2D Laplace problems. All the aforementioned methods have the common feature of evaluating the singular kernel functions in an indirect numerical methodology. The merits and drawbacks of the above-mentioned methods over the traditional MFS for solving elliptic boundary value problems are thoroughly discussed in Ref. [11].

This paper focuses on a recent technique, called the singular boundary method (SBM) [11–14]. The key idea in this method is to introduce the concept of the *origin intensity factor* to isolate the singularity of the fundamental solutions, and then to develop an inverse interpolation technique to determine the origin intensity factors from both the fundamental solution and its derivatives. This method is accurate and stable but amounts to solving the problems twice and thus the total computational cost is higher than the cost of other meshless boundary collocation methods, such as, the MFS.

The aim of the present paper is to present an alternative strategy for the direct evaluation of origin intensity factors in the SBM formulation. In the boundary element community, a significant contribution to the direct numerical evaluation of Cauchy principal value (CPV) integrals was proposed by Guiggiani and his collaborators [15–19]. It was shown that the general CPV integrals can be easily reduced to regular ones with simple manipulations, and the procedure can be applied whatever the type and order of the shape functions involved. Inspired by this work, we propose an improved SBM formulation based on the fact that the SBM and the indirect BEM have an underlying relationship. By analogy with the BEM, we develop a desingularization strategy for the direct

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efficient computation of singular kernels in the SBM without losing its meshless, integration-free, and easy-to-implement properties. It is important to stress that the SBM is still truly meshless and is different from the BEM in that it does not require numerical integration.

In addition, the SBM suffers from the so-called *boundary layer effect* [20–22] which results in poor approximations to the solution at points close to the boundary of the domain of the problem under consideration. In such cases, the kernel functions are "nearly" singular in the sense that the calculation point is close to, but not on, the boundary. The numerical evaluation of nearly singular integrals has already been investigated and researched in the BEM community [20,21,23–26]. However, as shown in Ref. [27], almost all these methods are limited to nearly singular integrals defined on low-order geometry elements. In a more recent study [14,28], the authors proposed an efficient non-linear coordinate transformation, based on sinh function [20], for the calculation of nearly singular integrals over curved geometry elements. In this study, this transformation is introduced for evaluating the nearly singular terms, which arise in the solution of 2D elasticity problems, using the proposed SBM formulation.

A brief outline of the rest of this paper is as follows. In Section 2, we describe the traditional SBM formulation for the solution of 2D elasticity problems. The proposed SBM formulation and its implementation are presented in Section 3. Section 4 introduces a non-linear transformation to remove the near singularities of the fundamental solutions. In Section 5, the accuracy and stability of the proposed SBM schemes are tested to three benchmark elasticity problems in which the SBM solutions are compared with the MFS and the BEM. Finally, some conclusions and remarks are provided in Section 6.

2. Traditional SBM formulation for 2D elasticity problems

In the absence of body forces, the equilibrium equations for the plane strain elastostatic problem, also known as the Navier equations, with respect to the displacements $u_i(\boldsymbol{x})$, $i = 1, 2$, can be stated as

$$\left\{2\frac{1-\mu}{1-2\mu}\right\}\frac{\partial^2 u_1(\boldsymbol{x})}{\partial x_1^2}+\frac{\partial^2 u_1(\boldsymbol{x})}{\partial x_2^2}+\left\{\frac{1}{1-2\mu}\right\}\frac{\partial^2 u_2(\boldsymbol{x})}{\partial x_1 \partial x_2}=0, \quad \boldsymbol{x}\in\Omega, \tag{1}$$

$$\left\{\frac{1}{1-2\mu}\right\}\frac{\partial^2 u_1(\boldsymbol{x})}{\partial x_1 \partial x_2}+\frac{\partial^2 u_2(\boldsymbol{x})}{\partial x_1^2}+\left\{2\frac{1-\mu}{1-2\mu}\right\}\frac{\partial^2 u_2(\boldsymbol{x})}{\partial x_2^2}=0, \quad x\in\Omega, \tag{2}$$

subject to the boundary conditions

$$u_i(\boldsymbol{x})=\bar{u}_i, \quad \boldsymbol{x}\in\Gamma_u(\textit{Dirichlet boundary conditions}), \tag{3}$$

$$t_i(\boldsymbol{x})=\bar{t}_i, \quad \boldsymbol{x}\in\Gamma_t(\textit{Neumann boundary conditions}) \tag{4}$$

where μ is Poisson's ratio, $t_i(x)$ denotes the component of the boundary traction in the ith coordinate direction, Γ_u and Γ_t comprise the whole boundary of the domain Ω as well as the exterior domain Ω^e as shown in Fig. 1, $\bar{u}_i$ and $\bar{t}_i$ represent the prescribed displacements and tractions, respectively.

The strains $\varepsilon_{ij}(\boldsymbol{x})$, $i, j = 1, 2$, are related to the displacement gradients by the kinematic relations

$$\varepsilon_{ij}(\boldsymbol{x})=\frac{1}{2}\left\{\frac{\partial u_i(\boldsymbol{x})}{\partial x_j}+\frac{\partial u_j(\boldsymbol{x})}{\partial x_i}\right\}, \tag{5}$$

and the stresses $\sigma_{ij}(\boldsymbol{x})$, $i, j, = 1, 2$, are related to the strains through Hooke's law by

$$\sigma_{ij}(\boldsymbol{x})=2G\left(\varepsilon_{ij}(\boldsymbol{x})+\frac{\mu}{1-2\mu}\varepsilon_{kk}(\boldsymbol{x})\delta_{ij}\right), \tag{6}$$

where δ_{ij} is the Kronecker delta and G is the shear modulus. The customary standard Cartesian notation for summation over repeated subscripts is used.

The boundary tractions $t_i(\boldsymbol{x})$, $i = 1, 2$, are defined in terms of the stresses as

$$t_i(\boldsymbol{x})=\sigma_{ij}(\boldsymbol{x})n_j(\boldsymbol{x}), \quad \boldsymbol{x}\in\Gamma, \tag{7}$$

where $n_j(\boldsymbol{x})$ is the direction cosine of the unit outward normal vector at the boundary point $\boldsymbol{x}$.

Employing indicial notation for the coordinates of the points $\boldsymbol{x}$ and $\boldsymbol{y}$, i.e., x_1, x_2 and y_1, y_2, respectively, the Kelvin fundamental solutions of the systems (1) and (2) can be expressed as [29]

$$U_{ij}(\boldsymbol{y},\boldsymbol{x})=-\frac{1}{8\pi G(1-\mu)}\{(3-4\mu)\ln r\delta_{ij}-\boldsymbol{r}_{,i}\boldsymbol{r}_{,j}\},(i,j=1,2), \tag{8}$$

where r is the Euclidean distance between $\boldsymbol{x}$ and $\boldsymbol{y}$, $r_{,i} = (y_i - x_i)/r$ denotes the derivatives of the distance r with respect to y_i.

The fundamental solution of the tractions can be obtained by first calculating the fundamental solutions of strains and then applying Hooke's law

$$T_{ij}(\boldsymbol{y},\boldsymbol{x})=-\frac{1}{4\pi(1-\mu)r}\{[(1-2\mu)\delta_{ij}+2r_{,i}r_{,j}]r_{,n} + (1-2\mu)(r_{,i}n_j(\boldsymbol{y})-r_{,j}n_i(\boldsymbol{y}))\}, \tag{9}$$

where $r_{,n} = r_{,i}n_i(y)$ represents the derivative of r in the direction of the outward normal at the point $\boldsymbol{y}$. Similarly, the fundamental solution of the stress is given as follows

$$D_{ijk}(\boldsymbol{y},\boldsymbol{x})=\frac{1}{4\pi(1-\mu)r}\{(1-2\mu)[r_{,k}\delta_{ij}-r_{,j}\delta_{ik}-r_{,i}\delta_{jk}]-2r_{,i}r_{,j}r_{,k}\}, \tag{10}$$

In the traditional MFS [30–32], the displacements and tractions can be approximated by a linear combination of fundamental solutions with respect to different source points $\boldsymbol{x}$ as follows:

$$u_i(\boldsymbol{y}^m)=\sum_{n=1}^{N}\alpha_j^n U_{ij}(\boldsymbol{y}^m,\boldsymbol{x}^n) =\sum_{n=1}^{N}[\alpha_1^n U_{i1}(\boldsymbol{y}^m,\boldsymbol{x}^n)+\alpha_2^n U_{i2}(\boldsymbol{y}^m,\boldsymbol{x}^n)], \tag{11}$$

$$t_i(\boldsymbol{y}^m)=\sum_{n=1}^{N}\alpha_j^n T_{ij}(\boldsymbol{y}^m,\boldsymbol{x}^n)=\sum_{n=1}^{N}[\alpha_1^n T_{i1}(\boldsymbol{y}^m,\boldsymbol{x}^n)+\alpha_2^n T_{i2}(\boldsymbol{y}^m,\boldsymbol{x}^n)], \tag{12}$$

where i, $j = 1, 2$, $\{\alpha_j^n\}_{n=1}^N$ represent the unknown coefficients, $\boldsymbol{y}^m \in \bar{\Omega} = \Omega \bigcup \partial\Omega$ is the mth collocation point, $\boldsymbol{x}^n$ stands for the nth source point, which lies outside $\bar{\Omega}$ (see Fig. 1(a) and (b)). The MFS requires a fictitious boundary outside the problem domain for the placement of the source points $\{\alpha^n\}_{n=1}^N$ to avoid the singularity of the fundamental solutions. However, despite many years of great effort, the placement of the distance between the real boundary and the fictitious boundary is based on experience and therefore troublesome, especially for problems in complicated geometries and higher dimensions [33,34].

The SBM also uses the fundamental solution as the basis function of its approximation. In contrast to the MFS, the collocation and source points of the SBM are coincident and are placed on the real boundary without using a fictitious boundary (see Fig. 1(c) and (d)). The basic idea of this method is to introduce the concept of the origin intensity factor to isolate the singularity of the fundamental solutions, so that the source points can be placed on the real boundary directly. With this idea in mind we represent the SBM interpolation as [11]