Computers and Structures 135 (2014) 155–168

Contents lists available at ScienceDirect

Computers and Structures

journal homepage: www.elsevier.com/locate/compstruc

Accurate approximations of concrete creep compliance functions based on continuous retardation spectra

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ARTICLE INFO

Article history:
Received 10 July 2013
Accepted 15 January 2014
Available online 10 February 2014

Keywords:
Concrete
Creep
Compliance function
Retardation spectrum
Dirichlet series
Laplace transform

ABSTRACT

This paper presents a detailed analysis of the continuous retardation spectra corresponding to a number of concrete creep models stipulated by various codes and recommendations. Approximations of various orders based on the Post–Widder formula are constructed, and the accuracy of the corresponding Dirichlet series approximating the compliance function is assessed. It is shown that the accuracy can often be substantially increased by appropriate modifications of the discrete retardation times used by the Dirichlet series. Practical hints regarding the choice of the discrete retardation times and the order of integration used in the evaluation of the compliance coefficients are provided.

1. Introduction

At sufficiently low stress levels, creep of concrete can be described by models formulated within the framework of linear viscoelasticity with aging. Once the principle of superposition is accepted, the material behavior is uniquely described by the compliance function or, alternatively, by the relaxation function. The compliance function reflects the time evolution of strain in a creep test at the unit stress level, while the relaxation function reflects the time evolution of stress in a relaxation test at the unit strain level. The uniaxial stress–strain relation can be written in one of the following equivalent forms:

$$\sigma(t) = \int_0^t R(t,t')\ \mathrm{d}\varepsilon(t') \tag{1}$$

$$\varepsilon(t) = \int_0^t J(t,t')\ \mathrm{d}\sigma(t') \tag{2}$$

Here, times t and t' correspond to the age of concrete, measured from the initial setting time, i.e., from the time when the hardening fresh concrete first becomes a solid (typically several hours after mixing), and therefore the stress and strain are not defined for negative times. The integrals are understood in the Stieltjes sense, so that they can be evaluated even for discontinuous stress and strain evolutions.

If the material point remains stress-free up to some initial time t_1, at which a stress σ_1 is applied abruptly and after which the stress evolves as a differentiable function of time, Eq. (2) can be rewritten in terms of the usual Riemann integral as

$$\varepsilon(t) = \sigma_1 J(t,t_1) + \int_{t_1}^t J(t,t')\dot{\sigma}(t')\ \mathrm{d}t' \tag{3}$$

where $\dot{\sigma}$ denotes the stress rate. Generalization to multiaxial stress states is straightforward under the assumptions of isotropic behavior and constant Poisson ratio. The structure of Eq. (3) is then preserved, with scalars ε and σ replaced by column matrices of strain and stress components, and with the right-hand side multiplied from the left by the elastic compliance matrix corresponding to the unit value of Young's modulus. The compliance function retains its scalar character.

Since creep tests are much more common in practice than relaxation tests, all design codes as well as academic concrete creep models consider the compliance function as the primary characteristic and describe it by a suitable formula with several parameters that can be calibrated by fitting of experimental data or estimated using empirical formulae (which usually take into account the concrete mix composition, curing time and conditions, member size and shape, and ambient relative humidity).

Analytical evaluation of the integral in (3) is possible only for simple models and simple stress histories. For general applications, numerical integration schemes are needed. If the integral in (3) is directly replaced by a finite sum [1], the memory requirements and number of arithmetic operations quickly grow with increasing

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http://dx.doi.org/10.1016/j.compstruc.2014.01.024
0045-7949/

number of computational time steps. For the evaluation of the creep strain increment based on such a direct integration scheme, the values of stress increments in all previous time steps must be known. In very large structural systems, the storage of the entire stress history at each integration point of each finite element and the evaluation of history integrals can become prohibitively expensive, even on modern powerful computers.

The computational complexity of the problem can be substantially reduced if the integral stress–strain relation is replaced by a differential one, dealing with certain history (internal) variables. The key idea is that a general compliance function (non-aging or aging) can be approximated by a Dirichlet series corresponding to a Kelvin rheological chain. Each unit of the chain is described by a differential equation that can be efficiently integrated in a step-by-step manner, as was first proposed in [2] for non-aging viscoelasticity and later extended in [3] to aging viscoelasticity. In contrast to the integral approach, it is not necessary to store the entire previous history but only a limited (and fixed) number of history variables that are updated after each step. The number of numerical operations per step is constant, independent of the total number of steps.

One unit of a Kelvin chain consists of a linear elastic spring and a linear viscous dashpot, coupled in parallel. In the case of non-aging viscoelasticity, its compliance function is given by

$$J(t,t') = \frac{1}{E}\left(1 - \mathrm{e}^{-(t-t')/\tau}\right) H(t-t') \tag{4}$$

where E is the spring stiffness (modulus), $\tau = \eta/E$ is the retardation time, η is the viscosity of the dashpot, and H is the Heaviside step function. The right-hand side of (4) depends on the time lag $t - t'$ only, rather than on the times t and t' separately. This is typical of non-aging materials. By coupling several Kelvin units in series, we obtain a Kelvin chain with compliance function in the form

$$J(t,t') \equiv \Phi(t-t') = \left[\frac{1}{E_0} + \sum_{\mu=1}^{M} \frac{1}{E_\mu}\left(1 - \mathrm{e}^{-(t-t')/\tau_\mu}\right)\right] H(t-t') \tag{5}$$

where E_μ, η_μ and $\tau_\mu = \eta_\mu/E_\mu$ are respectively the moduli, viscosities and retardation times of individual units $\mu = 1, 2 \ldots M$, and E_0 is the modulus of an elastic spring that is used as the "zeroth" unit of the chain and represents instantaneous elasticity. For convenience, we have introduced a new symbol Φ, denoting the compliance function considered as a function of a single variable—the time lag $t - t'$.

For deeper understanding of the behavior of a viscoelastic material, especially with regard to the time scale at which the viscous processes take place, the concept of retardation spectrum turns out to be useful. One Kelvin unit has a well-defined characteristic time, the retardation time $\tau = \eta/E$, which sets the intrinsic time scale of the model and determines which loading rates are considered as "slow" and which as "fast". The retardation time τ and the compliance $1/E$ are parameters of the compliance function (4) and uniquely characterize the model. A Kelvin chain can be characterized by the retardation times of individual units, τ_μ, and the corresponding compliances, $1/E_\mu, \mu = 1, 2, \ldots M$, and by the instantaneous compliance $1/E_0$. A general viscoelastic model can be characterized by a continuous spectrum of retardation times and compliances. From the mathematical point of view, such a spectrum is related to the inverse Laplace transform of the compliance function.

In a similar spirit, one could define the relaxation spectrum, which would be related to the inverse Laplace transform of the relaxation function, and for a Maxwell chain (obtained by coupling in parallel several Maxwell units, each of which consists of a spring and a dashpot coupled in series) would be discrete. As already mentioned, models for creep of concrete used in practice specify the compliance function and not the relaxation function, and so we will restrict our attention to the retardation spectrum.

The notion of retardation spectrum was introduced into concrete creep modeling in [4] and the influence of the specific choice of discrete retardation times was discussed in [5]. Continuous retardation spectra for the solidification theory of concrete creep were studied in [6,7]. If the limitations on available computational resources are not a major issue, extremely broad discrete spectra can be used [8]. In this paper we demonstrate that the efficiency and accuracy can be improved by a careful choice of the approximation method, taking into account the specific properties of various types of concrete creep models. The general technique presented in Section 2 will be applied to the log-power law in Section 3, to the ACI and CEB models in Section 4, to the drying creep compliance function of the B3 model in Section 5, to the JSCE model in Section 6, and finally to the new *fib* model in Section 7.

2. Relation between compliance function and retardation spectrum

Consider the Dirichlet series in (5), representing the compliance function of a non-aging Kelvin chain. The diagram of the compliances $1/E_\mu$ versus $\ln \tau_\mu$ is called the retardation spectrum of the material. For a Kelvin chain model with a finite number M of Kelvin units, the spectrum is discrete, consisting of a set of vertical lines (Fig. 1(a)). However, it is advantageous to conceive a generalization of Eq. (5) in which the spectrum is continuous (Fig. 1b), that is, the chain consists of infinitely many Kelvin units with infinitely small compliances $1/E_\mu$ and with the retardation times τ_μ distributed infinitely closely. According to this generalization, well known from classical viscoelasticity (e.g. [9]), one has, as the limit case of (5),

$$\Phi(t) = \int_{\tau=0}^{\infty} L(\tau)\left(1 - \mathrm{e}^{-t/\tau}\right) \mathrm{d}(\ln \tau), \quad t \geqslant 0 \tag{6}$$

in which function $L(\tau)$ characterizes the continuous spectrum. It should be noted that if L is a regular function (without any Dirac-like terms), $\Phi(t)$ is continuous and vanishes at $t = 0$. Therefore, the instantaneous compliance is not reflected in the spectrum and must be added as a separate constant $1/E_0$, which is of course straightforward.

The continuous spectrum is very useful when a given compliance function (e.g., defined by an analytical formula containing power functions and logarithms) is to be approximated by a Dirichlet series, which is needed for an efficient rate-type numerical approach. Of course, one could try to construct the approximation directly, by collocation [10] or by minimizing a suitable measure of the difference between the "exact" compliance function and the Dirichlet series. This is a somewhat tedious procedure, which can be circumvented by considering the Dirichlet series (5) as a numerical approximation of the integral in (6). If function $L(\tau)$ is known and the discrete retardation times τ_μ are selected, determination of the compliances $1/E_\mu$ is straightforward.

An important point is that a good approximation of the continuous spectrum can be obtained analytically, exploiting the Post–Widder formula [11,12] for the inversion of Laplace transform. It can be shown that the sequence of approximations

$$L_k(\tau) = -\frac{(-k\tau)^k}{(k-1)!}\, \Phi^{(k)}(k\tau), \quad k = 1, 2, \ldots \tag{7}$$

converges to the continuous spectrum, i.e.,

$$L(\tau) = \lim_{k\to\infty} L_k(\tau) \tag{8}$$

Here, k is the desired order of approximation, and $\Phi^{(k)}$ denotes the kth derivative of the compliance function.

Formula (7) can be derived by differentiation of Eq. (6). Taking the kth derivative with respect to t, we obtain