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On the analysis of thin-walled beams based on Hamiltonian formalism

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ABSTRACT

In this paper, the Hamiltonian approach developed for beam with solid cross-section is generalized to deal with beams consisting of thin-walled panels. The governing equations of plates and cylindrical shells for the panels are cast into Hamiltonian canonical equations and closed-form central and extremity solutions are found. Typically, the end-effect zones for thin-walled beams are much larger than those for beams with solid cross-sections. Consequently, extremity solutions affect the solution significantly. Correct boundary conditions based on the weak form formulation are derived. Numerical examples are presented to demonstrate the capabilities of the analysis. Predictions are found to be in good agreement with those of plate and shell FEM analysis.

1. Introduction

Thin-walled beams may be represented as an assembly of flat strips, or cylindrical shells, or both. Typically three length scales are involved: the wall thickness, which is far smaller than a representative dimension of the cross-section, the representative dimension of the cross-section, which is far smaller than the beam's length, and the beam's length. Due to these geometric characteristics, the deformation of thin-walled beams under load differs from that observed for beams with solid cross-sections. Under torsional loading, thin-walled beams may exhibit significant warping. Furthermore, if the cross-section is restrained from warping, axial and shear stresses develop, a phenomenon known as *constrained* or *nonuniform torsion*.

Thin-walled beam theories reduce the analysis of two-dimensional structures to one-dimensional problems that are far simpler to solve. Although more sophisticated formulations, such as those based on two-dimensional plate or shell finite element or finite strip methods, are able to capture the in-plane and out-of-plane warping behavior of thin-walled beams to the desired level of accuracy, the associated computational burden is often too heavy. Moreover, two-dimensional approaches do not provide an intuitive interpretation of the observed phenomena. The main goal of thin-walled beam theories is to approximate the assembly of two-dimensional plate and shell structures with one-dimensional models, while retaining an accurate representation of the local stress and strain fields over the contour of the cross-section.

Classical thin-walled beam theories have been proposed by Vlasov [1], Benscoter [2], Gjelsvik [3], and Shakourzadeh et al. [4] for isotropic thin-walled beams with open and closed cross-sections, respectively, based on kinematic assumptions. In many applications, thin-walled beams are, in fact, complex built-up structures with layers of anisotropic material stacked through the thickness of the walls. This new type of structural component prompted the development of new thin-walled beam theories [5–7]. With the goal of capturing the intricate stress field that develops under load, further refinements then followed by introducing more cross-section warping modes [8]. Although these approaches lead to higher accuracy, the number of unknowns increases considerably; furthermore, the identification of the dominant modes is often arduous.

Efficient thin-walled beam models can be obtained more rigorously from two-dimensional plate and shell equations through dimensional reduction techniques that split the original problem into a one-dimensional analysis over the beam's span and a one-dimensional cross-sectional analysis along the section's contour. These approaches can handle thin-walled beams made of anisotropic composite materials without increasing the total number of unknowns.

Asymptotic and multiscale analysis methods have been the tools of choice for dimensional reduction. Berdichevsky [9] proposed the Variational Asymptotic Method (VAM), in which asymptotic analysis is applied to the energy functional. Classical and Reissner thin-walled composite beam models based on VAM were developed by Hodges et al. [10,11].

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It is not necessary to use asymptotic methods to tackle thin-walled beam problems. Beam equations can be obtained directly through a separation of variables approach over the span-wise and along the section's contour directions. An approach of this type is the Generalized Beam Theory (GBT) proposed by Schardt [12] and further developed by numerous authors [13–18], among others. The formulations based on GBT provide a general procedure to determine the section's deformation modes. Projection of the governing equations onto these deformation modes leads to a unified and efficient one-dimensional beam formulation. This approach has been extended to buckling analysis (second-order GBT) and post-buckling analysis (third-order GBT).

A fundamental challenge in dimensional reduction of beam problems is to identify the algebraic structure of the cross-section's deformation modes and the characteristics of the corresponding solutions. Giavotto et al. [19] identified two types of solutions for beam problems: the central solutions, which are the solutions of Saint-Venant's problem, and the extremity solutions, which decay exponentially away from the beam's ends.

Mielke [20,21] found that the solutions of Saint-Venant's problem correspond to the center manifold of the system, which is spanned by the twelve generalized eigenvectors associated with four distinct Jordan chains. Of these twelve generalized eigenvectors, six correspond to the beam's rigid-body modes while the others six are the fundamental deformation modes of the beam: extension, torsion, and bending and shearing in two directions.

Zhong [22] developed novel analytical techniques based on the Hamiltonian formalism. A Hamiltonian operator characterizes the stiffness of the structure and its null and purely imaginary eigenvalues give rise to the solution of Saint-Venant's problem. The eigenvalues presenting a non-vanishing real part give rise to decaying solutions. As previously stated by Mielke, Zhong also identified the Jordan chains associated with the eigenvalues of the Hamiltonian operator with a vanishing real part.

Bauchau and Han [23] developed a three-dimensional beam theory based on the Hamiltonian formalism. The approach proceeds through a set of structure-preserving transformations using symplectic matrices and decomposes the solution into its central and extremity components. The same authors further generalized the approach to initially curved beams undergoing large motion but small strains [24], and helicoidal beams subjected to distributed loads [25]. For beams with solid cross-sections, the non-vanishing eigenvalues are associated with very small end effect zones near the beam's ends [25].

In this paper, the Hamiltonian formalism is extended to thin-walled beam problems. Governing equations of plates and shells for the panels are cast into Hamiltonian canonical equations and closed-form central and extremity solutions are found based on a procedure similar to that used for beams with solid-section. Typically, the end-effect zones for thin-walled beams are much larger than those for beams with solid cross-sections. Consequently, the extremity solutions should be taken into account because they alter the solution over the entire span of the beam. The other factor that makes extremity solutions important is the boundary condition. When subjected to twisting or shearing, thin-walled beams warp significantly. Far away from the end conditions, this warping if free to develop, but it is not consistent with built-in boundary conditions, a phenomenon known as "constrained warping effects." Saint-Venant's solution, which consists of the central solutions only, cannot predict constrained warping effects. Extremity solutions must be considered to satisfy the boundary constraints. In this paper, the boundary conditions are enforced in a weak sense, leading to a set of over-determined equations. Accurate predictions are obtained by combining central and extremity solutions, the latter are excited by the boundary conditions.

The following assumptions are made: (1) the straight thin-walled beam is an assembly of plates, or cylindrical shells, or both; (2) cross-sections of arbitrary geometry and material properties (heterogeneous and anisotropic) are considered, but remain uniform along the span; (3) strains and warping displacements remain small. Due to these assumptions, the governing equations of the problem can be cast into a homogenous Hamiltonian system with constant coefficients.

The paper is organized as follows: the kinematics of the problem and the governing equations of thin-walled beam problems are presented in Sections 2 and 3, respectively. The algebraic structure of the solutions is the focus of Section 4. The appropriate boundary conditions are derived in Section 5 and numerical examples are presented in the last section.

2. Kinematics of the problem

Fig. 1 depicts a straight thin-walled beam consisting of an assembly of plates and cylindrical shells, each of which is referred to as a panel. The beam's sectional contour, Γ, is the intersection of the cross-sectional plane with the panels' mid-planes. Let curve $\Gamma^{(i)}$ denoted the contour of the ith panel. The straight reference line of the beam is denoted $\mathcal{C}$. Consider an arbitrary point **B**, located at the intersection of the sectional plane with the reference line. Denote $\underline{r}_B$ the position vector of point **B** with respect to the origin of the inertial frame, $\mathcal{F} = [\mathbf{O}, \mathcal{I} = (\bar{\imath}_1, \bar{\imath}_2, \bar{\imath}_3)]$, and α_1 the arc-length coordinate along $\mathcal{C}$. The unit tangent vector to curve $\mathcal{C}$ is $\bar{b}_1 = \partial \underline{r}_B / \partial \alpha_1$. Because the plane of the cross-section is perpendicular to unit vector $\bar{b}_1$, a frame $\mathcal{F}_B = [\mathbf{B}, \mathcal{B}^* = (\bar{b}_1, \bar{b}_2, \bar{b}_3)]$ can be introduced which defines the position and orientation of the cross-section in the reference configuration, as illustrated in Fig. 1. Basis $\mathcal{B}^*$ is referred to as sectional basis. In the sequel, notation $(\cdot)^*$ indicates tensor components resolved in the sectional basis $\mathcal{B}^*$.

2.1. Strain components

Consider an arbitrary material point of the beam, **P**, located on contour $\Gamma^{(i)}$. Let $\underline{q}$ denote the relative position of point **P** with respect to point **B** and $\alpha_2^{(i)}$ the curvilinear coordinate measuring the arc length along curve $\Gamma^{(i)}$. A local basis, $\mathcal{B}^+ = (\bar{b}_1, \bar{t}^{(i)}, \bar{n}^{(i)})$, is introduced, where $\bar{t}^{(i)}$ and $\bar{n}^{(i)}$ are the unit tangent and normal

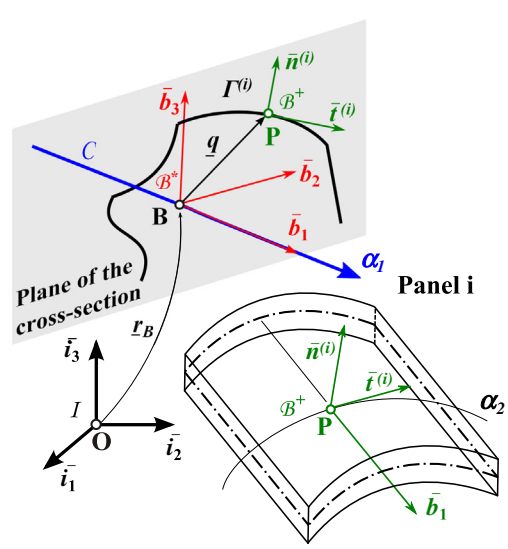

Fig. 1. Configuration of a beam with thin-walled section.