Journal of Economic Dynamics & Control 69 (2016) 436–459

Contents lists available at ScienceDirect

Journal of Economic Dynamics & Control

journal homepage: www.elsevier.com/locate/jedc

Envelope condition method with an application to default risk models

Cristina Arellano, Lilia Maliar*, Serguei Maliar, Viktor Tsyrennikov

Office 249, Department of Economics, Stanford, CA 94305-6072, USA

ARTICLE INFO

Article history:
Received 15 February 2013
Received in revised form
3 May 2015
Accepted 19 May 2016
Available online 8 June 2016

JEL classification:
C6
C61
C63
C68

Keywords:
Dynamic programming
Bellman equation
Endogenous grid
Curse of dimensionality
Large scale
Default risk

ABSTRACT

We develop an *envelope condition method* (ECM) for dynamic programming problems – a tractable alternative to expensive conventional value function iteration (VFI). ECM has two novel features: first, to reduce the cost of iteration on Bellman equation, ECM constructs policy functions using envelope conditions which are simpler to analyze numerically than first-order conditions. Second, to increase the accuracy of solutions, ECM solves for derivatives of value function jointly with value function itself. We complement ECM with other computational techniques that are suitable for high-dimensional problems, such as simulation-based grids, monomial integration rules and derivative-free solvers. The resulting value-iterative ECM method can accurately solve models with at least up to 20 state variables and can successfully compete in accuracy and speed with state-of-the-art Euler equation methods. We also use ECM to solve a challenging default risk model with a kink in value and policy functions.

1. Introduction

We develop an *envelope condition method* (ECM) for dynamic programming problems – a tractable alternative to expensive conventional value function iteration (VFI). ECM has two novel features: first, to reduce the cost of iteration on Bellman equation, ECM constructs policy functions using envelope conditions which are simpler to analyze numerically than first-order conditions. Second, to increase the accuracy of solutions, ECM solves for derivatives of value function jointly with value function itself. We complement ECM with other computational techniques that are suitable for high-dimensional problems, such as simulation-based grids, monomial integration rules and derivative-free solvers. The resulting value-iterative ECM method can accurately solve models with at least up to 20 state variables and can successfully compete in accuracy and speed with state-of-the-art Euler equation methods. We finally use ECM to solve a challenging default risk model with a kink in value and policy functions.

* Corresponding author. Tel.: +1 6507259069.
E-mail address: maliarl@stanford.edu (L. Maliar).

http://dx.doi.org/10.1016/j.jedc.2016.05.016
0165-1889/

We present ECM in the context of three applications: a one-agent growth model, a multi-country model of international trade and a default risk model. In our first application, we consider a stylized optimal growth model with inelastic labor supply. To solve such a model, conventional VFI constructs policy function by finding a maximum of the right side of the Bellman equation. This is done either directly (by using numerical maximization) or via first-order condition (by using a numerical solver). In contrast, the ECM methods construct policy functions by finding a solution to the envelope condition. Because such a solution can be derived in a closed form, ECM requires only direct calculations and avoids the need of either numerical optimization or numerical solvers when iterating on the Bellman equation.

We also develop a version of ECM that approximates derivatives of value function (possibly, jointly with value function) instead of value function itself. This version of ECM produces more accurate solutions than an otherwise identical ECM that solves exclusively for value function. This is because solving accurately for value function does not necessarily leads to sufficiently accurate approximations of its derivatives. For example, if value function is approximated with polynomial of degree n, then its derivatives are effectively approximated with polynomial of degree $n-1$, i.e., we "lose" one polynomial degree when differentiating value function. In contrast, by approximating derivatives of value function directly, we focus on the object that identifies policy functions and hence, obtain more accurate solutions.

We then investigate the convergence properties of the constructed class of ECM methods in the context of the studied optimal growth model. We establish that ECM has the same fixed point solution as the regular Bellman operator with some additional technical restriction. However, the ECM operator does not possess the property of contraction mapping like the regular Bellman operator. In this respect, the ECM class of methods is similar to the Euler equation class of methods for which global convergence results are generally infeasible. Nevertheless, the fact that the convergence theorems cannot be established for some numerical methods does not mean that the method is not useful. In particular, Euler equation methods are useful in a variety of contexts. In our numerical experiments, the ECM method has good convergence properties and produces accurate solutions in a wide range of the model's parameters.

In our second application, we construct a version of ECM that is suitable for high-dimensional applications, including stochastic simulation, non-product monomial integration rules, and derivative-free solvers, to solve a multicountry growth model with up to 10 countries (20 state variables).[1] This model is the one studied in the February 2011's special issue of the Journal of Economic Dynamics and Control (henceforth, JEDC project) which compares the performance of six state-of-the-art solution methods.[2] We show that the ECM methods is tractable and reliable in this setting and is able to successfully compete with state-of-the-art Euler equation methods in the high-dimensional applications which were part of the JEDC project. For our most accurate third-degree polynomial solutions, maximum unit-free residuals in the model's equations are always smaller than 0.002% on a stochastic simulation of 10,000 observations.

Finally, our third application is a default risk model of Arellano (2008). Default models are challenging computationally because value and policy functions have kinks and the price function of debt depends on the level of debt reflecting default probabilities. Nonetheless, at the optimal debt level, the decision functions are continuously differentiable and satisfy FOCs; see Clausen and Strub (2013) for a version of the envelope theorem that applies to models with default risk and a survey of envelope theorems in the literature. We show that the ECM methods are fast in computing this model. Relative to the expensive VFI method, ECM speeds up the computation time by more than 50 times. However, the convergence is more difficult to attain in this model. Numerical errors in approximating value function along iteration may lead to nonmonotone policy functions and result in non-convergence. Damping and shape preserving restrictions on value function can help to deal with this problem.

While our analysis is limited to the benchmark default risk model, we think that ECM can be useful for many other applications with default risk. In fact, a substantial hurdle for the growing literature on sovereign default is the computational cost; see, e.g., Aguiar and Gopinath (2006), Chatterjee et al. (2007), Hopenhayn and Werning (2008), Bianchi et al. (2009), Maliar et al. (2008), Chatterjee and Eyigundor (2011), Arellano et al. (2013), Tsyrennikov (2013), and Aguiar et al. (2015); see Aguiar and Amador (2013) for a review of the literature on sovereign debt. The ECM methods can facilitate the development of this literature by expanding the types of problems that can be efficiently solved.

We next discuss the relation of the ECM method to other numerical methods in the literature. Dynamic programming methods are introduced in Bellman (1957) and Howard (1960) in the context of stationary, infinite-horizon Markovian problems. There is a large body of literature that focuses on solving DP problems including methods based on discretization of state space (e.g., Rust, 1996, 1997), stochastic simulation methods (e.g., Smith, 1991, 1993), Maliar and Maliar, 2005), learning methods (e.g., Bertsekas and Tsitsiklis, 1996), perturbation methods (e.g., Judd, 1998), policy iteration (e.g., Santos and Rust, 2008), nonexpensive approximations (Stachurski, 2008), approximate DP methods (e.g., Powell, 2011), polyhedral approximations (e.g., Fukushima and Waki, 2011), random contractions (Pal and Stachurski, 2013); also see Rust (2008), Judd (1998), Santos (1999), and Stachurski (2009) for literature reviews. From one side, many methods, which are accurate and reliable in problems with low dimensionality, are intractable in problems with high dimensionality. This is in particular true

[1] See Maliar and Maliar (2014) for a survey of these and other numerical techniques that are tractable in problems with a large number of state variables.

[2] The objectives of the JEDC project are described in Den Haan et al. (2011); the methodology of the numerical analysis is outlined in Juillard and Villemot (2011); the results of the comparison analysis are provided in Kollmann et al. (2011b). The six participating methods are first- and second-order perturbation methods of Kollmann et al. (2011a), stochastic simulation and cluster-grid algorithms of Judd et al. (2011), monomial rule Galerkin method of Pichler (2011) and Smolyak's collocation method of Malin et al. (2011).