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The method of endogenous gridpoints in theory and practice ☆

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ABSTRACT

The method of endogenous gridpoints (ENDG) significantly speeds up the solution to dynamic stochastic optimization problems with continuous state and control variables by avoiding repeated computations of expected outcomes while searching for optimal policy functions. I provide an interpolation technique for non-rectilinear grids that allow ENDG to be used in n-dimensional problems in an intuitive and computationally efficient way: the acceleration of ENDG with non-linear grid interpolation is nearly constant in the density of the grid. Further, ENDG has only been shown by example and has never been formally characterized. Using a theoretical framework for dynamic stochastic optimization problems, I formalize the method of endogenous gridpoints and present conditions for the class of models for which it can be used.

1. Introduction

The traditional solution method for a dynamic stochastic optimization problem is for the researcher to choose a set of gridpoints in the state space, solve for the optimal action at each of these points, and then generate an approximation to the true policy function through interpolation. Because next period's state is subject to risk and there is usually no closed form solution, this method involves extremely costly root-finding with repeated computation of numeric integrals. The method of endogenous gridpoints (ENDG), first proffered by Carroll (2006),[1] provides an alternative approach that inverts the problem: rather than asking what the optimizing agent should *do now* if he begins the period in state x, the method of endogenous gridpoints asks what an agent ending the period in state z must *have just done* to find himself there, assuming he acted optimally to arrive at z. In this way, the pre-decision state space points are solved endogenously rather than exogenously fixed in advance. The method minimizes the number of numeric integrals needed to solve the problem and often avoids root-finding entirely, greatly accelerating the solution.

However, practical application of ENDG to models with multiple endogenous state dimensions and multiple controls has proved to be problematic. As the policy functions are non-linear, the method produces an irregular array of pre-decision

☆ This paper combines two previous drafts: "Endogenous Gridpoints in Multiple Dimensions: Interpolation on Non-Linear Grids" and "A General Theory of the Method of Endogenous Gridpoints".

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[1] Originally applied to a problem with a one dimensional state space and one continuous control variable, ENDG has since been extended multiple times: Barillas and Fernández-Villaverde (2007) describe how to accommodate multiple controls, Fella (2014) allows non-concave value functions, Hintermaier and Koeniger (2010) adjust the method for occasionally binding constraints, while Iskhakov et al. (2012) develop a method that allows one continuous and one discrete control variable.

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state space points rather than a rectilinear grid that can be used with standard linear interpolation. Moreover, the method has only been shown by example and never fully characterized in a general setting – the conditions under which ENDG can be used have never been formally stated. To address these theoretical and practical holes in the literature, this paper provides two main contributions.

First, I present in Section 3 a technique for interpolating on an irregular array of gridpoints that exploit known relationships among the points: the endogenous pre-decision gridpoints are *ordered* identically to the exogenous post-decision gridpoints that generated them. In two dimensions, the pre-decision gridpoints thus subdivide the state space into a grid of irregular quadrilateral sectors, each of which can be continuously mapped to the unit square to allow standard bilinear interpolation. This technique avoids the long construction or evaluation procedures used in other interpolation methods for irregular arrays. Section 4 demonstrates the computational gains of ENDG in combination with "non-linear grid interpolation" using a two dimensional (2D) model introduced in Section 2.

Second, I formalize the method of endogenous gridpoints in a general framework for dynamic stochastic optimization problems, providing sufficient conditions for its application. Presented and discussed in Section 5, these conditions primarily concern the decomposition of the state transition function into "intraperiod" and "interperiod" components and the accompanying existence of a "post-decision state" that serves as a sufficient statistic of both the state and control variables.

2. Benchmark model

To motivate the need for an alternative interpolation technique, this section presents a model with two endogenous state variables and two controls. For ease of reference, the model is a slight extension of Ludwig and Schön (2014), adding additional interperiod risk.[2]

2.1. Statement of model

The individual is a finitely lived lifetime expected utility maximizer whose state at discrete time t is characterized by his money resources (previous wealth plus current income) m_t and his health capital h_t. He derives a flow of utility from consumption c_t via CRRA utility function $u(c_t)$ with coefficient ρ, and discounts future utility at a rate of β per period. At the beginning of period t, the individual receives labor income of $\omega_t h_t$, where wage rate ω_t is distributed log-normally $\mathcal{N}(\overline{\omega}/(1-\mho),\sigma_\omega^2)$, with a probability $\mho$ point mass at $\omega_t=0$ representing unemployment. The individual must choose his levels of consumption c_t and investment in health capital i_t so that $m_t-c_t-i_t\geq 0$. Investment generates additional health capital according to production function $f(i_t)=(\gamma/\alpha)i_t^\alpha$.

Any unspent resources accumulate at gross interest factor R, so that next period's resources are $m_{t+1}=R(m_t-c_t-i_t)+\omega_{t+1}h_{t+1}$. Health capital depreciates from one period to the next at depreciation rate $\delta_{t+1}\sim U[\overline{\delta}-\sigma_\delta,\overline{\delta}+\sigma_\delta]$, and so next period's capital is $h_{t+1}=(1-\delta_{t+1})(h_t+f(i_t))$. The individual thus faces both permanent (via δ) and transitory (via ω) income risk. Moreover, he faces a mortality risk based on his health capital, with the probability of survival into period $t+1$ given by $s(h_{t+1})=1-\phi/(1+h_{t+1})$; assume there is a terminal period T beyond which the individual cannot live. If the individual does not survive, he receives no more income, nor can he consume and derive utility.

The agent's problem in any non-terminal[3] period can be written in Bellman form as

$$V_t(m_t,h_t)=\max_{c_t,i_t} u(c_t)+\beta\int s(h_{t+1})V_{t+1}(m_{t+1},h_{t+1})\,dF(\omega,\delta) \tag{1}$$

$$\text{s.t. } m_t-c_t-i_t\geq 0,\quad m_{t+1}=R(m_t-c_t-i_t)+\omega h_{t+1},\quad h_{t+1}=(1-\delta)(h_t+f(i_t)).$$

The optimal policy functions for c_t and i_t are defined by replacing max with arg max in (1).

2.2. Solving the model with ENDG

To solve for optimal behavior using ENDG, begin with the first order conditions of the model:

$$u'(c_t)=\beta R\int s(h_{t+1})V_{t+1}^m(m_{t+1},h_{t+1})\,dF(\omega,\delta),$$

$$f'(i_t)=\frac{R\int s(\cdot)V_{t+1}^m(\cdot)dF(\omega,\delta)}{\int(1-\delta)\left[s'(\cdot)V_{t+1}(\cdot)+s(\cdot)\left(\omega V_{t+1}^m(\cdot)+V_{t+1}^h(\cdot)\right)\right]dF(\omega,\delta)} \tag{2}$$

[2] Risk generalizes the model and eliminates the need for a constrained optimization routine, maintaining focus on ENDG and non-linear grid interpolation rather than the complications of dual search routines.

[3] The solution in the terminal period is trivial, yielding a value function of $V_T(m,h)=u(m)$ and marginal value functions of $V_T^m(m,h)=u'(m)$ and $V_T^h(m,h)=0$; policy functions are $c_T(m,h)=m$ and $i_T(m,h)=0$.