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# Robust measurement of (heavy-tailed) risks: Theory and implementation

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ABSTRACT

Every model presents an approximation of reality and thus modeling inevitably implies model risk. We quantify model risk in a non-parametric way, i.e., in terms of the divergence from a so-called nominal model. Worst-case risk is defined as the maximal risk among all models within a given divergence ball. We derive several new results on how different divergence measures affect the worst case. Moreover, we present a novel, empirical way built on model confidence sets (MCS) for choosing the radius of the divergence ball around the nominal model, i.e., for calibrating the amount of model risk. We demonstrate the implications of heavy-tailed risks for the choice of the divergence measure and the empirical divergence estimation. For heavy-tailed risks, the simulation of the worst-case distribution is numerically intricate. We present a Sequential Monte Carlo algorithm which is suitable for this task. An extended practical example, assessing the robustness of a hedging strategy, illustrates our approach.



## 1. Introduction

Financial market crises, increasingly complex product designs, and demanding regulatory rules highlight the importance of an efficient and effective risk management. At the core of every risk management process is the qualitative and quantitative assessment of outstanding risks. However, risk management is not riskless in itself. Any quantitative risk assessment must rely on modeling assumptions and is thus prone to model risk. No risk management process can afford to ignore uncertainties about models and their parameters. Therefore, a reliable risk management process must also aim at revealing and quantifying these risks. This is the main goal of this paper.

Our paper belongs to the literature which incorporates the “divergence approach” to model-robustness into financial risk management (Friedman, 2002b; Glasserman and Xu, 2014; Breuer and Csiszár, 2012; Ben-Tal et al., 2013). This approach originates from the operations research and control literature, e.g. Whittle (1990). To economists it is best known through the work of Hansen, Sargent and coauthors in macroeconomics, see e.g. Hansen and Sargent (2011).[1] The idea behind this “divergence approach” is to assess worst-case values of target quantities by perturbing a nominal/reference model in a non-

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[1] A second well-known and related stream of the literature is the theory of ambiguity which goes back to Gilboa and Schmeidler (1989). Maccheroni et al. (2006) establishes, among others, the connection between this literature and the Hansen–Sargent approach.

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parametric way. Concretely, the worst case is taken over all alternative models whose divergence (in some predetermined sense) from the nominal model lies within a given radius. Robust decision making can thus be interpreted as a two-player game between the decision maker and a fictitious adversary who tries to harm him as much as possible by manipulating the model – within the confines of a known upper bound on model risk.

Our main objective is to give risk-takers a guidance for assessing how much of a threat model risk is to their risk management process. This is somewhat complimentary to the literature on robust risk measures (Föllmer and Schied, 2011). Their main aim is to find risk measures which have desirable properties from a regulatory perspective. That is, they focus on developing robust risk measures rather than on the implementation of a robust risk management process.

The following two questions are the focal points of the previous literature: (1) Given a nominal model, a divergence measure and a target quantity, how can we explicitly characterize the worst-case value of the target quantity? (2) How does the answer to question (1) influence optimal decisions? The present paper centers around three questions which inevitably arise if one wishes to implement the robustness approach in practice: (3) How does the choice of the divergence measure influence the worst-case analysis? (4) How can we meaningfully bound the amount of model risk by estimating the divergence between our model and the "true" data-generating process? (5) How can we calculate worst-case quantities in a numerically reliable way? In particular, one of our main goals is simply to highlight the problems that arise when making the divergence approach to model risk operational. While we provide a complete set of solutions to these problems, we are well aware that any reader will find some of these solutions more convincing than others.

There are good answers to questions (1) and (2) in the literature. For this reason, we build on previous results for question (1), e.g. Ben-Tal et al. (2013) and Breuer and Csiszár (2013). For simplicity, we abstract from issues of optimal control, and thus from question (2), and concentrate on the assessment of worst-case risk. In a sense, question (2) is the final question: An answer only becomes truly useful once there are answers to the remaining four questions. Indeed our results can easily be combined with the good answers to that question found in the literature.[2]

Question (3) is particularly relevant for the heavy-tailed risks frequently encountered in financial applications. The so-called *f*-divergences possess many desirable properties with regard to applications within the robustness approach (Ben-Tal et al., 2013; Breuer and Csiszár, 2013). However, the impact of the actual choice of *f*-divergence measure on the set of potential worst-case distributions has not been studied yet. In the context of model misspecification, this is particularly important as the nominal model and the true data generating process can differ considerably. Under heavy tails, a "standard" implementation of the robustness approach, relying on Kullback–Leibler (KL) divergence as a divergence measure, often leads to infinite worst-case values. For this reason Glasserman and Xu (2014) propose to use polynomial divergences in heavy-tailed applications. In practice, the choice of the divergence measure, however, influences the worst-case analysis to a great extent. We show that polynomial divergence places rather strict (but opaque) restrictions on the set of possible worst-case models. We demonstrate through a number of examples that working with polynomial divergence is essentially equivalent to postulating that the true risk is not much more heavy-tailed than the nominal model. The essence is that rare disasters which are an important motivation of any worst-case analysis are excluded from the model uncertainty. Thus, this postulate is at odds with a robust risk management approach. In addition, the choice of the divergence measure strongly affects the possibilities for measuring and bounding model risk.

Question (4) is, in a way, the most important one: unless we can convincingly declare an upper bound on model risk, i.e., on the divergence between the nominal model and the data, the worst-case is arbitrarily bad – regardless of how well we can answer questions (1)–(3) and (5). In the literature, the issue of specifying the amount of model risk has largely been ignored. Exceptions are Hansen and Sargent (2011) and Ben-Tal et al. (2013) who both focus on sampling inaccuracy rather than model misspecification. For instance, the proposal of Hansen and Sargent (2011) is shaped by the challenges of macroeconomic modeling where a time-series of interest may be observed only once every few months. In marked contrast, there is an abundance of data in many financial applications. Indeed, there is by now something of a dichotomy in financial modeling: there are models which are good at mimicking the data (say, sophisticated GARCH models) and there are models which are tractable enough to be preferred as a back-up for decisions (say, the Black–Scholes model). Thus, our answer to (4) is the following: estimate the divergence between the nominal model and the model(s) mimicking the data best in a statistically reliable way. We propose to define the *best* model(s) in terms of the model confidence set approach of Hansen et al. (2011). The divergence can be estimated relying on *k*-nearest-neighbor-estimators for KL- and polynomial divergences (Pérez-Cruz, 2008; Póczos and Schneider, 2011). Empirical divergence estimation, however, can only yield reasonable results if the divergence between the nominal model and the alternative(s) is finite. We show that for natural financial applications this is violated by polynomial divergence. Therefore, we propose to guarantee the finiteness of worst-case values and divergence estimators by combining KL-divergence with *truncated* risk functionals (Cont et al., 2010).

In many challenging financial applications, we have to rely on Monte Carlo methods to obtain the worst-case distribution and related quantities. A simple numerical scheme which is proposed in previous literature is Importance Sampling (Frideman, 2002a,b; Glasserman and Xu, 2014). However, we demonstrate that Importance Sampling fails to converge in some natural applications. Thus, the numerical problem at hand is more difficult than one might think, and standard Monte Carlo algorithms should be treated with caution. Yet there seems to be little point in conducting a worst-case analysis based on a

[2] In a way, our answers to the remaining questions can be seen as inevitable steps in the practical implementation of a broad literature in mathematical finance which studies decision problems under robustness, see e.g., Bordigoni et al. (2007).