Journal of Economic Dynamics & Control 58 (2015) 250–264

Contents lists available at ScienceDirect

Journal of Economic Dynamics & Control

journal homepage: www.elsevier.com/locate/jedc

Superhedging under ratio constraint

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ARTICLE INFO

Article history:
Received 13 January 2015
Received in revised form
15 May 2015
Accepted 25 June 2015
Available online 2 July 2015

JEL classification:
C63
G13

Keywords:
Superhedging
Ratio constraint
European option
Asian option

ABSTRACT

We consider superhedging of contingent claims under ratio constraint. It has been widely recognized that the minimum cost of superhedging a contingent claim with certain portfolio constraints is equal to the price of a claim with appropriately modified payoff but without constraints. In terms of the backward stochastic differential equation (BSDE) and the variational inequality equation approach, we revisit this result and provide two counterexamples.

1. Introduction

Since the pioneering work of Black and Scholes (1973) and Merton (1973) in the complete market, much research has focused on option pricing in the incomplete market where a perfect hedging is no longer available. It turns out that in an incomplete market, a range for the rational prices of a contingent claim exists and the upper bound of the rational price range (upper price hereafter) corresponds to the minimum cost of superhedging the contingent claim.

Portfolio hedging in an incomplete market can be risky, and the regulatory constraint might be enforced. There is an extensive literature on the superhedging problem with portfolio constraints (e.g., Cvitanic and Karatzas, 1993; Naik and Uppal, 1994; Bardhan, 1995; El Karoui and Quenez, 1995; Karatzas and Kou, 1996, 1998; Broadie et al., 1998; Cuoco and Liu, 2000; Huang, 2002; Schmock et al., 2002). In particular, Broadie et al. (1998) show that the smallest superhedging cost of a contingent claim with certain constraints is equal to the price of a dominating claim without constraints, and the result remains valid for a variety of options, including vanilla options, multi-asset options, barrier options, and lookback options. They also point out that a general result for path-dependent options seems impossible and analysis has to be done on a case-by-case basis.

The dominating claim solution proposed by Broadie et al. (1998) significantly simplifies the problem in terms of a minor change of the existing Black–Scholes pricing system when a certain portfolio constraint is enforced. It is of practical interest to investigate how robust this approach is. In this paper, we consider two scenarios: vanilla options with time-varying ratio

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[1] Dai acknowledges support from Singapore AcRF Grants R-146-000-188/201-112 and NUS Global Asia Institute - LCF Fund R-146-000-160-646.
[2] Xu acknowledges the support from National Scientific Foundation of China, No. 11371350.

http://dx.doi.org/10.1016/j.jedc.2015.06.009
0165-1889/

constraint, and Asian options with constant ratio constraint. We are interested in the two scenarios because time-varying risk preference of an agency may result in time-varying portfolio constraint, and Asian options are among the most actively traded OTC derivatives with strong path dependence. We aim to resolve this problem in terms of the variational inequality equation approach. We also provide simple criterions to examine the result of Broadie et al. (1998).

As in Peng (1999), we formulate the superhedging problem as a backward stochastic differential equation (BSDE). Using the relation between BSDEs and partial different equations, we derive a variational inequality equation with gradient constraints that the upper price function of a constrained contingent claim satisfies. It is interesting that the variational inequality equation resembles the governing equation for the utility maximization problem with transaction costs which has been widely studied (e.g., Davis and Norman, 1990; Shreve and Soner, 1994; Dai and Yi, 2009; Dai et al., 2009; Kallsen and Muhle-Karbe, 2010). We use the idea of Dai and Yi (2009) to obtain an equivalent obstacle problem through which it is easy to identify whether results analogous to Broadie et al. (1998) are valid or not. It is worthwhile pointing out that our approach is easy to implement and can be used to price a large class of contingent claims, including path-dependent options. In particular, we find two counterexamples against the results of Broadie et al. (1998): the European vanilla option with time-varying ratio constraint and the fixed strike arithmetic Asian option with constant ratio constraint. In these cases, one cannot simply follow Broadie et al. (1998) and find a dominating claim to compute the superhedging cost. Instead, a free boundary problem must be involved to determine the superhedging cost and characterize the superhedging strategy.

Cuoco and Liu (2000) examine optimal consumption and investment choices and the cost of hedging contingent claims in the presence of margin requirements. A margin requirement can be regarded as a ratio constraint provided that only long positions in the risky asset are involved. However, when a portfolio consists of short positions, the loss of interest on the proceeds from short sales leads to nonlinear budget constraints. It should be pointed out that our paper does not touch nonlinear constraints.

The rest of the paper is organized as follows. In Section 2, we introduce the model setup and formulate the problem as a constrained BSDE through which we present a sufficient condition for the existence of the upper price and the minimal superhedging strategy. In Section 3, we consider the European call option under time-varying ratio constraint. We first present the variational inequality equation for the upper price and then study it through an equivalent obstacle problem which is easy to investigate. Section 4 is devoted to the arithmetic Asian call option under constant ratio constraint. We conclude in Section 5. All proofs are placed in the Appendix.

2. Superhedging under ratio constraint

2.1. The financial model

We consider a financial market which comprises a risk-free money market account and a risky underlying asset (e.g., stock or foreign currency). Their prices B_t and S_t are assumed to follow the equations

$$dB_t = r_t B_t dt, \quad B_0 = B, \tag{2.1}$$

$$dS_t = (\mu_t - q_t) S_t dt + \sigma_t S_t dW_t, \quad S_0 = x, \tag{2.2}$$

where $r_t \geq 0$ is the risk-free rate, μ_t, q_t and σ_t are the cum-dividend expected return rate of, the dividend yield of, and the volatility of the stock, respectively, and W_t is a standard one dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. $\mathbf{P}$ is known as the *objective* probability measure, and the information structure is given by a right-continuous filtration $(\mathcal{F}_t : 0 \leq t \leq T)$, where $\mathcal{F}_t$ is the σ-algebra generated by $(W_s)_{0 \leq s \leq t}$ and augmented by all $\mathbf{P}$-null set in $\mathcal{F}$. We assume that r_t, μ_t, q_t, and σ_t are all deterministic functions of time.

We consider a hedger who aims at hedging her short position in a contingent claim. Assume that the hedger is a price taker whose actions have no impact on market prices. The hedger can decide, at any time $t \in [0, T]$, her cumulative consumption C_t and the amount π_t of the wealth V_t invested in the risky asset, where the processes π_t and $V_t - \pi s_t$ are predictable, and C_t is a predictable and non-decreasing process with $C_0 = 0$.

A self-financing portfolio strategy is a triplet (V, π, C) such that

$$V_t = V_0 + \int_0^t (V_t - \pi_t) \frac{dB_t}{B_t} + \int_0^t \pi_t \frac{dS_t}{S_t} + \int_0^t \pi_t q_t \, dt - C_t,$$

or equivalently,

$$dV_t = r_t V_t \, dt - dC_t + \pi_t \sigma_t [dW_t + \theta_t \, dt],$$

where $\theta_t = (\mu_t - r_t)/\sigma_t$ is the market price of risk associated with the uncertainty induced by W_t. The strategy is called feasible if $\int_0^T |\sigma_t \pi_t|^2 \, dt < +\infty$, $\mathbf{P}$ a.s.

For later use, we denote $\mathbf{Q}$ as the *risk neutral* probability measure induced by the Radon–Nikodym derivative process:

$$\frac{d\mathbf{Q}}{d\mathbf{P}}\Big|_{\mathcal{F}_t} = \exp\left(- \int_0^t \theta_s dW_s - \frac{1}{2} \int_0^t \theta_s^2 \, ds \right), \quad 0 \leq t \leq T.$$