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A simple method for computing equilibria when asset markets are incomplete

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ABSTRACT

The problem of computing equilibria for general equilibrium models with incomplete real asset markets, or GEI models for the sake of brevity, is reconsidered. It is shown here that the rank-dropping behavior of the asset return matrix could be dealt with in rather a simple fashion: We first compute its singular value decomposition, and then, through this decomposition, construct, by the introduction of a homotopy parameter, a new matrix such that it has constant rank before a desired equilibrium is reached. By adjunction of this idea to the homotopy method, a simpler constructive proof is obtained for the generic existence of GEI equilibria. For the purpose of computing these equilibria, from this constructive proof is then derived a path-following algorithm whose performance is finally demonstrated by means of three numerical examples.

1. Introduction

After the hard work of a number of researchers over a number of decades, the proposition that the actual market is incomplete has been widely acknowledged to be a fact in the town of economics. This then naturally raises the question of how much of the general equilibrium theory for complete markets could be extended to the case of incomplete markets. With this question in mind, the present paper is concerned with the general equilibrium model with incomplete real asset markets, and studies the existence and computation of its equilibrium.

There are various reasons for the market being incomplete, such as asymmetric information, moral hazard, and transaction costs (see Geanakoplos, 1990), and this incompleteness has both remarkable and far-reaching implications. Two of them, perhaps most notable of all, are that perfect competition need not necessarily lead to market efficiency, and that nominal assets are in marked contrast to real assets, in terms of their impact on the existence of equilibrium. As regards the latter, equilibria have been shown in Werner (1985) and Cass (2006) to exist when the assets marketed are nominal. But, on the other hand, it has been shown in Hart (1975), by way of example, that, when the assets marketed are real, equilibria may fail to exist for GEI models. This unpleasant phenomenon, fortunately, has turned out to be nongeneric, and the generic existence of GEI equilibria has been established in, for instance, Duffie and Shafer (1985), Geanakoplos and Shafer (1990), and Hirsch et al. (1990). The basic idea of these papers is to consider the demand function as defined on the Cartesian product of the price simplex and a Grassmann manifold, and then make extensive use of results from differential topology.

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In the same spirit, a procedure for computing GEI equilibria has been devised in Demarzo and Eaves (1996). The efficiency of this procedure however is adversely affected by the high-dimension of the Grassmann manifold.

The first constructive proof that dispenses with the use of Grassmann manifold is to our knowledge presented in Brown et al. (1996). In that paper the ingenious idea of switching homotopies is invented. Relatively speaking, the procedure derived from this idea for computing GEI equilibria is, as remarked in Demarzo and Eaves (1996), not robust and not easy to implement. Motivated presumably by these considerations, Schmedders (1998, 1999) introduces the idea of a penalty function with penalties imposed on transactions on the asset market. On the one hand, this method is intuitively appealing and becomes easy to implement, but, on the other hand, its generic convergence has not yet been established.

Confronted with this state of affairs we propose in the present paper to give another constructive proof for the generic existence of GEI equilibria. Recall that the main difficulty for so doing consists in the fact that the asset return matrix may drop rank as the price varies. To circumvent this our idea is rather simple, and could be intuitively described as follows. Let $R(p)$ be the asset return matrix, where p denotes the price vector. We first find its singular value decomposition, given say by $L_1(p)\Sigma(p)L_2(p)$, where $L_1(p)$ and $L_2(p)$ are two orthogonal matrices and $\Sigma(p)$ is a diagonal matrix. From matrix theory it follows that the diagonal entries of $\Sigma(p)$ are nonnegative, and the number of its strictly positive diagonal entries is precisely the rank of $R(p)$. Based on this fact, we then introduce a parameter t in $[0,1]$, add the term $1-t$ to every diagonal entry of $\Sigma(p)$, and let the resultant matrix be $\Sigma(p,t)$. Using it we form a new matrix $R(p,t)=L_1(p)\Sigma(p,t)L_2(p)$, which is easily seen to be of full rank for every t in $[0,1)$. With $R(p,t)$ as a substitute for $R(p)$, we are finally able, by means of the homotopy method, to establish the generic existence of GEI equilibria. This idea is carried out in detail in Section 3. It must be remarked that all the technicalities required to realize the idea have already been available in the literature, especially in Brown et al. (1996); what we have to do is simply put them in order.

We notice that from the constructive proof above a path-following algorithm can be derived for computing GEI equilibria. This algorithm is, in comparison with the existing ones (Brown et al., 1996; Schmedders, 1998), marked by its simplicity and low-dimensional search space. To test its performance we apply it in Section 4 to compute the equilibria for some numerical examples; the results of which turn out to be rather satisfactory, showing that the algorithm derived is both effective and efficient. Before embarking on an elaboration of the preceding results, we have to first of all spell out the model economy with which we shall be concerned; let us do this in the ensuing section.

2. The GEI model

We shall study the same general equilibrium model with incomplete real asset markets as for instance in Brown et al. (1996), Duffie and Shafer (1985), and Schmedders (1998). This model consists of two periods with uncertainty over the states of nature in the second period. In this period we assume that there are S number of possible states, which will be indexed by s running from 1 to S; call the first period state zero. In each sate there are L commodities available, which will be indexed by l running from 1 to L. We shall hereafter use the symbol, x_{sl}, to refer to a certain amount of commodity l in state s.

We assume K real assets marketed in the economy, which will be indexed by k running from 1 to K; to make the market incomplete we further assume $K<S$. By a real asset we shall mean an asset whose payoff in each state is specified by a bundle of commodities in that state. Let $M=L(S+1)$; we can therefore characterize each asset by a vector of dimension M whose first L components denote its payoff in state zero, the second L components its payoff in state one, and so forth. This in turn enables us to characterize the asset market by a matrix A, of dimension $M\times K$, with its k-column designating the payoff of asset k. Without loss of generality and to simplify the exposition we shall assume that every asset has a zero payoff in state zero.

Let $\Delta=\{p\in\mathbf{R}^M_+ | \sum_{i=1}^M p_i=1\}$ be the price simplex and Δ_{++} its relative interior. For each $p\in\Delta_{++}$ it is often convenient to partition it state-wise into $p=(\overline{p}_0,\dots,\overline{p}_S)$ with $\overline{p}_s\in\mathbb{R}^L$ being the price vector for state s, $s=0,\dots,S$. As in Brown et al. (1996) we define for a particular $p=(\overline{p}_0,\dots,\overline{p}_S)\in\Delta$:

$$P=\begin{bmatrix} 0 & \overline{p}_1 & 0 & \cdots & 0 \\ 0 & 0 & \overline{p}_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \overline{p}_S \end{bmatrix}_{S\times M};$$

the nominal return matrix at the price vector p of the assets is then given by

$$R(p)=PA\in\mathbf{R}^{S\times K}. \tag{1}$$

We assume $H+1$ households inhabiting in the economy, which will be indexed by h running from 0 to H. Household h is characterized by an endowment vector $e^h\in\mathbf{R}^M_{++}$ and a utility function $U_h:\mathbf{R}^M_{++}\to\mathbf{R}$ with the following standard properties:

(i) *smoothness*: U_h is C^∞,
(ii) *strict monotonicity*: $DU_h(x)\in\mathbf{R}^M_{++}$ for all x in $\mathbf{R}^M_{++}$,
(iii) *differentiably strict convexity*: $z^\top D^2U_h(x)z<0$ for all $z\neq 0$ with $DU_h(x)z=0$,
(iv) *boundary condition*: $\{x\in\mathbf{R}^M_{++} | U_h(x)\geq U_h(x_0)\}$ is closed in $\mathbf{R}^M$ for all x_0 in $\mathbf{R}^M_{++}$.