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Solving asset pricing models with stochastic volatility

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ABSTRACT

This paper provides a closed-form solution for the price-dividend ratio in a standard asset pricing model with stochastic volatility. The growth rate of the endowment is a first-order Gaussian autoregression, while the stochastic volatility innovations can be drawn from any distribution for which the moment-generating function exists. The solution is useful in allowing comparisons among numerical methods used to approximate the nontrivial closed form. The closed-form solution reveals that, when using perturbation methods around the deterministic steady state, the approximate solution needs to be sixth-order accurate in order for the parameter capturing the conditional standard deviation of the stochastic volatility process to be present.

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1. Introduction

Stochastic volatility has become an important feature of macroeconomic models that seek to explain both stylized business cycle and asset pricing facts. Since closed-form solutions elude richer macroeconomic models, various numerical methods have been proposed to provide an approximated solution. The contribution of this paper is to present a simple (asset pricing) stochastic volatility model in which an exact solution (for the price-dividend ratio) exists, which may serve as a benchmark from which to compare alternative numerical approximation methods.

Burnside (1998) provided an exact solution for the Lucas (1978) asset pricing model with Gaussian, autoregressive dividend growth shocks and time-separable constant relative risk-aversion (CRRA) preferences.[1] Bidarkota and McCulloch (2003) and Tsionas (2003) extended Burnside's solution to shocks with stable distributions and shocks with well-defined moment-generating functions (MGFs), respectively, while Chen et al. (2008) and Collard et al. (2006) extended it to the case with non-time-separable preferences through habits in consumption.[2] In each case, the solutions provide a useful benchmark against which to test numerical solution algorithms. This paper follows in that tradition. It extends the Burnside model by adding stochastic volatility to the dividend growth process.

Since Bansal and Yaron (2004) showed the importance of stochastic volatility to account for stylized asset pricing facts, the use of stochastic volatility has become a widespread addition to standard business cycle models. Yet, even beside the

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[1] An early contribution by Labadie (1989) also provided the solution in a slightly more general context.

[2] In related work, Calin et al. (2005) develop a method that finds the solutions for analytic utility functions that offer closed forms for a wide class of probability distributions for the state variable. Similarly, Le et al. (2010) extend the Gaussian dynamic term structure model to a larger class of MGFs.

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demand for business cycle models to match stylized asset pricing facts, there is a growing use of stochastic volatility in macromodelling. Stock and Watson (2002) and Sims and Zha (2006) are prominent examples arguing that time-varying volatility is important in accounting for the dynamics of U.S. aggregate data. Among Dynamic Stochastic General Equilibrium (DSGE) researchers, stochastic volatility is being put to many applications: Bloom et al. (2007) consider the role of time-varying uncertainty for investment dynamics, Justiniano and Primiceri (2008) investigate the sources of the Great Moderation, and Fernández-Villaverde et al. (2011) study the effects of stochastic volatility in fiscal shocks on economic activity, to name just a few.

Because of the increasing importance of stochastic volatility, which naturally adds additional nonlinearity into the solution of models, a growing literature has been testing how different numerical solution methods that solve equilibrium models with stochastic volatility perform. Caldara et al. (2012), for example, compare perturbation methods (of second and third order), Chebyshev polynomials, and value function iteration in a real business cycle model with stochastic volatility.

In this paper, I show the exact solution for the price-dividend ratio of a simple asset pricing model as a nontrivial function of the model's two state variables, the current dividend growth rate and the current volatility of the dividend growth process.[3] Innovations to the dividend growth rate are drawn from a Gaussian distribution. Innovations to the stochastic volatility process can be drawn from any distribution for which the MGF exists. For much of the paper, I follow Bansal and Yaron (2004) and assume Gaussian shocks for the stochastic volatility innovations. However, a gamma distribution is potentially appealing because it ensures that the realizations of the stochastic volatility process are strictly nonnegative and because it displays skewness and kurtosis.

The closed-form solution has the following properties: First, the price-dividend ratio increases when the volatility of dividend growth increases, as well as when the volatility of the stochastic volatility process increases. Second, the sensitivity of the price-dividend ratio to a change in the volatility state is increasing in the persistence of the stochastic volatility process. I derive an expression for the unconditional mean of the price-dividend process, as well as several other endogenous objects of interest, such as the risk-free rate and the conditional equity risk premium. Since the closed-form solution for the price-dividend ratio takes the form of an infinite sum, I provide parameter conditions under which the price-dividend ratio (and its unconditional mean) are finite. I also show where to truncate the infinite summation when calculating the solution numerically to ensure that the truncation error is no larger than a given value with a given probability.

Finally, I show how two alternative low-order polynomial approximation techniques perform in terms of numerical accuracy: (1) a first-order approximation following Campbell and Shiller (1988) that exploits the normality of the stochastic processes; and (2) the perturbation method around a deterministic steady state, popular among macro-DSGE researchers. I find two results of note: First, a fourth-order perturbation is required to generate a similar order of accuracy close to the steady state as the approximation that exploits the normality of the stochastic processes. Second, a sixth-order perturbation approximation is required for the parameter capturing the conditional standard deviation of the stochastic volatility process to show up in the approximated solution.

The rest of the paper is structured as follows. Section 2 presents the basic asset pricing model with stochastic volatility, and Section 3 presents the general closed-form solution. Section 4 applies the model and further discusses its uses. Section 5 concludes. The appendix provides derivations of the paper's key results, while an extensive online appendix provides additional detail, describes a variant of the basic model, and tests the model's asset pricing implications.

2. The asset pricing model

There is a representative agent who maximizes the expected discounted stream of utility

$$\mathbf{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}, \tag{1}$$

subject to the budget constraint:

$$c_t + s_{t+1} p_t \le (d_t + p_t) s_t, \tag{2}$$

where $\mathbf{E}_t$ is the mathematical expectations operator conditional on the time t information set, c_t is consumption, and s_t denotes units of an asset whose price at date t is p_t with dividends d_t. The discount factor is $\beta \in (0,1)$, and the coefficient of relative risk aversion is $\gamma > 0$ and $\gamma \neq 1$. The growth rate of dividends, denoted $x_t \equiv \log(d_t/d_{t-1})$, is assumed to follow a Gaussian $AR(1)$ process:

$$x_t = x + \rho(x_{t-1} - x) + \sqrt{\eta_t}\varepsilon_t, \tag{3}$$

where x is the steady-state growth rate of dividends, $\rho \in (-1,1)$ is the persistence parameter, and ε_t is a sequence of independently and identically distributed (i.i.d.) innovations from the standard normal distribution. The innovations to x_t are scaled by $\sqrt{\eta_t}$. η_t is therefore the conditional variance of dividend growth and is time varying. In particular, it follows an

[3] The model features CRRA preferences and not recursive preferences as in Bansal and Yaron (2004), which means that the model does not solve the risk-free rate and equity premium puzzles (see Mehra and Prescott, 1985; Weil, 1989). However, this feature does not diminish the model's usefulness as a testing ground for numerical solution methods interested in capturing the effects of stochastic volatility.