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# Turnpike property and convergence rate for an investment model with general utility functions

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ABSTRACT

In this paper we aim to address two questions faced by a long-term investor with a power-type utility at high levels of wealth: one is whether the turnpike property still holds for a general utility that is not necessarily differentiable or strictly concave, the other is whether the error and the convergence rate of the turnpike property can be estimated. We give positive answers to both questions. To achieve these results, we first show that there is a classical solution to the HJB equation and give a representation of the solution in terms of the dual function of the solution to the dual HJB equation. We demonstrate the usefulness of that representation with some nontrivial examples that would be difficult to solve with the trial and error method. We then combine the dual method and the partial differential equation method to give a direct proof to the turnpike property and to estimate the error and the convergence rate of the optimal policy when the utility function is continuously differentiable and strictly concave. We finally relax the conditions of the utility function and provide some sufficient conditions that guarantee the turnpike property and the convergence rate in terms of both primal and dual utility functions.


## 1. Introduction

The turnpike property is a classical problem in financial economics and has been discussed by many researchers for both discrete time and continuous time models, see Back et al. (1999) and Huang and Zariphopoulou (1999) for exposition and literature. It is well known that the optimal proportion of wealth invested in the risky asset for a constant relative risk aversion utility is a constant. The turnpike property says the same trading strategy is approximately optimal at the beginning of the investment period for any utility function behaving asymptotically like a power utility, provided the investment horizon is sufficiently long. The economic intuition of this phenomenon is that "When the interest rate is strictly positive, the present value of any contingent claim having payoffs bounded from above can be made arbitrarily small when the investment horizon increases. Thus an investor concentrates his wealth in buying contingent claims that have payoffs unbounded from above at the very beginning of his horizon. As a consequence, it is the asymptotic property of his utility function as wealth goes to infinity that determines his optimal investment strategy at the very beginning of his horizon.", see Cox and Huang (1992).

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For the Merton problem with a power utility $x^p/p$, where $p<1$ is a constant and $x>0$ is the portfolio wealth, the optimal amount of investment in risky asset at time $t$ is given by $\theta x/(\sigma(1-p))$, a constant proportion of wealth, where $\theta$ is the Sharpe ratio and $\sigma$ the asset volatility. The optimal amount of investment in risky asset at time $t$ for a general utility $U$ is given by

$$A(\tau,x)=-\frac{\theta}{\sigma}\frac{u_x(\tau,x)}{u_{xx}(\tau,x)}, \tag{1.1}$$

where $\tau=T-t$ is the time to the investment horizon $T$ and $u$ is the value function that is a solution to a nonlinear partial differential equation (PDE) (see (3.2)) and the notation changes above it) with the initial condition $u(0,x)=U(x)$, provided that $u$ is continuously differentiable with respect to $\tau$ and $x$. We say the turnpike property holds if

$$\lim_{\tau\to\infty}A(\tau,x)=\frac{\theta}{\sigma(1-p)}x \tag{1.2}$$

for all $x>0$. The turnpike property (1.2) means that the optimal strategy for the utility $U$ is close to the Merton optimal strategy for the power utility at any level of the initial wealth $x$ as long as the time to horizon $\tau$ is sufficiently long. Having the turnpike property in portfolio management is highly desirable as it makes the investment decision process simple and efficient.

One of the standard assumptions in the study of the turnpike property in the literature is that the utility $U$ is continuously differentiable and strictly concave. Cox and Huang (1992) use the probabilistic method to show that the turnpike property holds if the inverse of the marginal utility $(U')^{-1}$ satisfies some conditions. Huang and Zariphopoulou (1999) establish the turnpike property with the viscosity solution method to the HJB equation when the marginal utility $U'$ behaves like that of a power utility at high levels of wealth and satisfies some other conditions. Jin (1998) discusses an optimal investment and consumption problem and shows the turnpike property holds in the sense of convergence on average ($L^1$ and $L^2$ convergence) when $(U')^{-1}$ is "regularly varying" at the origin, see aforementioned papers for details and the references therein for other models, mainly discrete time models.

Another noticeable missing feature in the literature is that there are no discussions on the convergence rate even if the turnpike property is known to hold, that is, if the following inequality holds

$$\left|A(\tau,x)-\frac{\theta}{\sigma(1-p)}x\right|\le D(x)e^{-c\tau} \tag{1.3}$$

for some positive constants $c$ and $D(x)$ (see (3.32)). The significance of (1.3) is that it gives the error estimate of the turnpike property and helps one to determine the length of the investment period in order to achieve the specified accuracy of replacing the optimal strategy with the Merton optimal strategy.

It is natural and interesting to ask if the turnpike property (1.2) still holds for general utilities (strictly increasing, continuous and concave, but not necessarily continuously differentiable and strictly concave) and if the error estimate (1.3) can be established and the convergence rate and the error magnitude can be computed. Our main contribution in this paper is to give positive answers to both questions. The error and convergence analysis with closed-form $c$ and $D(x)$ is the first in the literature in the study of the turnpike property, to the best of our knowledge. In the process of proving these results we show the existence of the classical solution to the HJB equation and find the representation of the solution in terms of the dual function, which is of independent interest and may be applied to solve many utility maximization problems with the stochastic control method.

The discussion of the turnpike property can be decomposed into two problems: one is a finite horizon utility maximization and the other the limiting process for the optimal strategies as the investment horizon tends to infinite. Stochastic control theory is one of the standard methods for utility maximization. It applies the dynamic programming principle and Ito's lemma to derive a nonlinear PDE, called the HJB equation, for the value function. If there is a smooth classical solution to the HJB equation one may use the verification theorem to find the value function and the optimal feedback control. For excellent expositions of stochastic control theory and its applications in utility maximization, see Fleming and Soner (1993) and Pham (2009) and references therein.

The smoothness of the value function is a highly desirable property as it naturally leads to a feedback optimal control in terms of the value function and its derivatives, which is especially relevant to the turnpike property as the limiting behavior of the optimal strategies is to be studied. However, one cannot expect to have a classical solution to the HJB equation unless some conditions are imposed, for example, the uniform ellipticity of the diffusion coefficient of wealth process, which is not satisfied in general. It is well known that the value function has a closed-form solution to the HJB equation for the power utility. When the utility is strictly concave, continuously differentiable, satisfying some growth conditions, and the trading constraint set is a closed convex cone, the value function is a classical solution to the HJB equation, see Karatzas and Shreve (1998). For a general continuous increasing concave utility $U$ satisfying $U(0)=0$ and $U(\infty)=\infty$, Bian et al. (2011) show that there exists a classical solution to the HJB equation and that the value function is smooth if an exponential moment condition is satisfied at the optimal control.

To study the turnpike property (1.2) and the convergence rate (1.3) we first extend the results of Bian et al. (2011) to more general utilities. We remove the condition $U(\infty)=\infty$ as we need to address the utility that behaves like the negative power utility for large wealth. We show that there exists a classical solution $w$ to the HJB equation and that $w$ has a representation

$$w(t,x)=v(t,y(t,x))+xy(t,x)$$