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The method of fundamental solutions for three-dimensional inverse geometric elasticity problems

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ABSTRACT

We investigate the numerical reconstruction of smooth star-shaped voids (rigid inclusions and cavities) which are compactly contained in a three-dimensional isotropic linear elastic medium from a single set of Cauchy data (i.e. nondestructive boundary displacement and traction measurements) on the accessible outer boundary. This inverse geometric problem in three-dimensional elasticity is approximated using the method of fundamental solutions (MFS). The parameters describing the boundary of the unknown void, its centre, and the contraction and dilation factors employed for selecting the fictitious surfaces where the MFS sources are to be positioned, are taken as unknowns of the problem. In this way, the original inverse geometric problem is reduced to finding the minimum of a nonlinear least-squares functional that measures the difference between the given and computed data, penalized with respect to both the MFS constants and the derivative of the radial coordinates describing the position of the star-shaped void. The interior source points are anchored and move with the void during the iterative reconstruction procedure. The feasibility of this new method is illustrated in several numerical examples.

1. Introduction

In direct problems in solid mechanics, one has to determine the response of a system when the governing partial differential equations (equilibrium equations), the constitutive and kinematics equations, the initial and boundary conditions for the displacement and/or traction vectors and the geometry of the domain occupied by the solid are all known. However, if at least one of the above conditions is partially or entirely lacking, then one has a so-called inverse problem. Moreover, it is well-known that inverse problems are in general unstable, in the sense that small measurement errors in the input data may amplify significantly the errors in the solution, see e.g. [16]. Such inverse problems have been extensively studied, both theoretically and numerically, over the last three decades and an overview of these developments can be found in [10].

In the case of inverse geometric problems in solid mechanics, which represent an important subclass of inverse problems, the geometry of the domain occupied by the solid is not entirely known, however some additional information is available. More specifically, part of the boundary of the solution domain is not known but either the displacements or the tractions are known on this portion, whilst the remaining boundary is known and both displacement and traction measurements are available on it. The inverse geometric problems described above can be further subdivided into two subclasses, depending on the location of the unknown boundary, namely (i) identification of an unknown boundary or corrosion-type problems (the unknown boundary is a part of the outer boundary of the solution domain), see e.g. [27–29], and (ii) identification of voids, i.e. cavities and rigid inclusions (the unknown boundary is an inner boundary), see e.g. [12,21–23].

There are important studies that are devoted to the latter subclass of inverse geometric problems in elasticity. Alessandrini et al. [1,2] proved that the volume (size) of a rigid inclusion in an elastic isotropic body can be estimated by an easily expressed quantity related to work, depending only on the boundary traction and displacement. Morassi and Rosset [33] provided upper and lower bounds on the size of unknown defects, such as cavities or rigid inclusions, in an elastic body, from boundary measurements of tractions and displacements. Later, they considered the inverse problem of determining a rigid inclusion inside an isotropic elastic body from a single set of Cauchy data on the outer boundary and proved its uniqueness and conditional stability [34]. The issue of uniqueness in determining cavities in a heterogeneous isotropic elastic medium in two dimensions was investigated by Ang et al.

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[4], who used the unique continuation for the isotropic Lamé system and geometric considerations. Ben Ameur et al. [8] developed a rather general approach for identifiability and local Lipschitz stability of cavities in two and three spatial dimensions in linear elasticity and thermo-elasticity. Ikehata and Itou [19] considered the reconstruction problem of an unknown polygonal cavity in a homogeneous isotropic elastic body and provided an extraction formula of the convex hull of the cavity using the enclosure method.

With respect to the numerical identification of voids in elasticity, most of the studies available in the literature are devoted to the two-dimensional case. A regularized boundary integral formulation for the detection of flaws in planar structural membranes from the displacement measurements given at some boundary locations and the applied loading was proposed in [9]. Hsieh and Mura [18] developed a combined boundary element method (BEM)-nonlinear optimization algorithm for the detection of both the location and the shape of an unknown cavity in an elastic medium. Mellings and Aliabadi [30] presented a dual boundary element formulation for the identification of the location and size of internal flaws in two-dimensional structures. Kassab et al. [24] and Ulrich et al. [37] investigated the non-destructive detection of internal cavities in the inverse elastostatic problem using the BEM. The level set method and a regularization technique related to the perimeter of the unknown inclusion were employed by Ben Ameur et al. [7] for the numerical reconstruction of a void from a single Cauchy data. We finally mention that some three-dimensional elastodynamic inverse problems have been solved using the BEM in [6,11].

In recent years the method of fundamental solutions (MFS), originally proposed by Kupradze and Aleksidze [26] and introduced as a numerical method by Mathon and Johnston [31], has been used extensively for the numerical solution of inverse and related problems primarily due to its ease of implementation. An extensive survey of the applications of the MFS to inverse problems is provided in [20]. It appears that the MFS was used for the first time for the solution of inverse geometric problems in linear elasticity by Alves and Martins [3], who adapted to the detection of rigid inclusions or cavities in an elastic body the method of Kirsch and Kress [25]. The method of [3] decomposes the inverse problem into a linear and ill-posed part in which a Cauchy problem is solved using the MFS and a nonlinear part in which the unknown boundary of the void is sought as a zero level set for a rigid inclusion (or computed iteratively, in an optimization scheme for a class of approximating shapes, for a cavity). In contrast to this, Karageorghis et al. [21] adopted a fully nonlinear MFS in which the nonlinear and ill-posed parts are dealt with simultaneously using a nonlinear regularized minimization. The reconstructions obtained using this latter method are more accurate than those obtained by decomposition methods, see e.g. [36].

The purpose of this paper is to extend to three-dimensional elasticity the two-dimensional analysis of [21], the same way we have done for the harmonic scalar case in [22,23]. In particular, we extend the work of [23] to three-dimensional inverse geometric problems, see also [12]. The paper is organized as follows: Section 2 is devoted to the mathematical formulation of the inverse geometric problem investigated. The MFS discretization for this problem is described in Section 3, whilst the implementational details are given in Section 4. In Section 5, we investigate four examples. Finally, some concluding remarks and possible future work are provided in Section 6.

2. The Cauchy–Navier equations of elasticity

2.1. The problem

We consider the boundary value problem in a bounded domain $\Omega \subset \mathbb{R}^3$ for the Cauchy–Navier system of elasticity for the displacement $\boldsymbol{u}$ in the form (see e.g. [17])

$$\mu\Delta\boldsymbol{u} + \frac{\mu}{1-2\nu}\nabla\cdot\nabla\boldsymbol{u} = \boldsymbol{0} \quad \text{in} \quad \Omega, \tag{1a}$$

where $\mu > 0$ is the shear modulus and $\nu \in (0, 1/2)$ is the Poisson ratio, subject to the Dirichlet boundary conditions

$$\boldsymbol{u} = \boldsymbol{f} \quad \text{on} \quad \partial\Omega_2, \tag{1b}$$

and the homogeneous boundary conditions

$$\alpha\boldsymbol{u} + (1-\alpha)\boldsymbol{t} = \boldsymbol{0} \quad \text{on} \quad \partial\Omega_1, \tag{1c}$$

where α is 0 or 1. The inverse problem we are concerned with consists of determining not only the displacement $\boldsymbol{u}$, but also the unknown inclusion Ω_1 so that $\boldsymbol{u}$ satisfies the Cauchy–Navier equations (1a), given the Dirichlet data $\boldsymbol{f}$ in (1b), the homogeneous boundary condition (1c) and the Neumann traction measurements

$$\boldsymbol{t} = \boldsymbol{g} \quad \text{on} \quad \partial\Omega_2. \tag{1d}$$

In the above, $\Omega = \Omega_2 \setminus \Omega_1$, where $\overline{\Omega}_1 \subset \Omega_2$, is a bounded annular domain with boundary $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$. The void Ω_1 may have many connected components, but Ω should be connected. Eq. (1c), covers both homogeneous Dirichlet ($\alpha = 1$, i.e. Ω_1 is a rigid inclusion) and Neumann ($\alpha = 0$, i.e. Ω_1 is a cavity) boundary conditions on $\partial\Omega_1$. In (1c), $\boldsymbol{t}$ represents the traction defined by

$$\boldsymbol{t} = \boldsymbol{\sigma}\boldsymbol{n} \quad \text{on} \quad \partial\Omega_2. \tag{2}$$

In (2), the outward normal unit vector to the boundary at the point (x_1, x_2, x_3) is denoted by $\boldsymbol{n}(x_1, x_2, x_3) = (n_{x_1}, n_{x_2}, n_{x_3})$, whilst $\boldsymbol{\sigma}$ is the stress tensor given, in terms of the strain tensor $\boldsymbol{\varepsilon} = \left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\mathsf{T}}\right)/2$, by Hooke's law [17], namely

$$\boldsymbol{\sigma} = 2\mu\left[\boldsymbol{\varepsilon} + \frac{\nu}{1-2\nu}\operatorname{tr}(\boldsymbol{\varepsilon})\boldsymbol{I}\right] \quad \text{in} \quad \overline{\Omega}, \tag{3}$$

where $\boldsymbol{I}$ is the 3×3 identity matrix.

If the Dirichlet and Neumann data (1b) and (1d) are not identically zero, then the uniqueness of the solution pair $(\boldsymbol{u}, \Omega_1)$ of the inverse problem (1a)–(1d) holds, see [3].

3. The method of fundamental solutions (MFS)

In the application of the MFS to (1), we seek an approximation to the solution of the three-dimensional Cauchy–Navier equations of elasticity as a linear combination of fundamental solutions in the form [35]

$$\begin{aligned}&\boldsymbol{u}_{NM}(\boldsymbol{a}^1, \boldsymbol{a}^2, \boldsymbol{b}^1, \boldsymbol{b}^2, \boldsymbol{c}^1, \boldsymbol{c}^2, \xi^1, \xi^2; \boldsymbol{x}) \\ &\quad = \sum_{s=1}^{2}\sum_{n=1}^{N}\sum_{m=1}^{M}\boldsymbol{G}(\xi^s_{n,m}, \boldsymbol{x})\left[a^s_{n,m}\ b^s_{n,m}\ c^s_{n,m}\right]^{\mathsf{T}},\end{aligned} \tag{4}$$

where $\boldsymbol{G}(\xi, \boldsymbol{x}) = \left[G_{ij}(\xi, \boldsymbol{x})\right]_{1\leqslant i,j\leqslant 3}$ is the fundamental solution matrix for the displacement vector in three-dimensional isotropic linear elasticity given by

$$\boldsymbol{G}(\xi, \boldsymbol{x}) = \frac{1}{16\pi\mu(1-\nu)}\frac{1}{|\boldsymbol{x}-\xi|}\left[(3-4\nu)\boldsymbol{I} + \frac{\boldsymbol{x}-\xi}{|\boldsymbol{x}-\xi|}\otimes\frac{\boldsymbol{x}-\xi}{|\boldsymbol{x}-\xi|}\right], \tag{5}$$

and the vectors $\boldsymbol{a}^s = \left[a^s_{1,1}, a^s_{1,2}, \ldots, a^s_{N,M}\right]$, $\boldsymbol{b}^s = \left[b^s_{1,1}, b^s_{1,2}, \ldots, b^s_{N,M}\right]$ and $\boldsymbol{c}^s = \left[c^s_{1,1}, c^s_{1,2}, \ldots, c^s_{N,M}\right]$, $s = 1, 2$, contain the unknown MFS coefficients. Similarly, from (2), (4) and (5), the tractions are approximated by [5]

$$\begin{aligned}&\boldsymbol{t}_{NM}(\boldsymbol{a}^1, \boldsymbol{a}^2, \boldsymbol{b}^1, \boldsymbol{b}^2, \boldsymbol{c}^1, \boldsymbol{c}^2, \xi^1, \xi^2; \boldsymbol{x}) \\ &\quad = \sum_{s=1}^{2}\sum_{n=1}^{N}\sum_{m=1}^{M}\boldsymbol{T}(\xi^s_{n,m}, \boldsymbol{x})\left[a^s_{n,m}\ b^s_{n,m}\ c^s_{n,m}\right]^{\mathsf{T}}\end{aligned} \tag{6}$$