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Location-scale portfolio selection with factor-recentered skew normal asset returns

Quan Gan [*]

Discipline of Finance, University of Sydney, Sydney 2006, NSW, Australia

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ABSTRACT

This paper analyzes the single period portfolio selection problem on the location-scale return family. The skew normal distribution, after recentering and reparameterization, is shown to be in this family. The recentered and reparameterized distribution, called factor-recentered skew normal, can be expressed as a skew factor model which is characterized by a location parameter and two scale parameters. Risk preference on scale parameter is non-monotonic and risk averse investors prefer larger (smaller) scale when the scale is negative (positive). The three-parameter efficient set is a part of conical surface bounded by two lines. Positive-skewness portfolios and negative-skewness portfolios do not coexist in the efficient set. Numerical cases under constant absolute risk aversion are analyzed with its closed-form certainty equivalent. An asset pricing formula which nests the CAPM is obtained.

1. Introduction

The mean–variance portfolio analysis developed by Markowitz (1952) is widely used in financial applications. Recent research extends Markowitz's work to incorporate information of higher moments in portfolio decisions. A popular approach is to analyze the efficient set in the moments (e.g. mean–variance-skewness) space. de Athayde and Flôres (2004) derive several general properties of the efficient set. Mencía and Sentana (2009) characterize the efficient set when asset returns follow a location-scale mixture of normals. Briec and Kerstens (2010) use a shortage function to search a direction to increase odd moments and decrease even moments.

This paper takes another approach which focuses on return distributions in the location-scale family. The location-scale distributions can be expressed as factor models which are of particular interest in financial modeling. Asymmetry in asset returns is introduced when at least one random factor follows an asymmetric distribution. Meyer (1987) analyzes the preferences over the two-parameter location-scale family. The normal distribution, for example, is in this family. Meyer's results establish the consistency of mean–variance analysis and expected utility maximization. Wong and Ma (2008) generalize Meyer's results to the location-scale family governed by multiple scale parameters. In earlier work, Simaan (1993) analyzes portfolio selection problem on a three-parameter location-scale family which represents a perturbation of elliptical

[*] Tel.: +61 2 91140831.
E-mail address: quan.gan@sydney.edu.au

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distributions. Among the three parameters, Simaan does not specify the risk preference on a scale parameter which captures asymmetry. This makes Simaan's work on efficient set incomplete.

A pertinent research question relatively overlooked by the literature is what distributions (especially skew distributions) are in the location-scale family. This question is equivalent to finding skew distributions that can be written as latent factor models. I fill this research gap by showing that the skew normal distribution (Azzalini, 1985; Azzalini and Dalla Valle, 1996; Azzalini and Capitanio, 1999) is in the three-parameter location-scale family. The statistical literature on this distribution is well developed.[1] Under its typical parameterization, however, it is difficult to find out that the skew normal distribution actually belongs to the location-scale family and can be subsequently expressed as a factor model. In this paper, I show that the skew normal distribution is in the location-scale family after recentering and reparameterization. The recentered and reparameterized distribution, called factor-recentered skew normal, can be expressed as a skew factor model and is characterized by a location parameter and two scale parameters. The new parameterization facilitates the analysis of preference on each parameter.

Asset returns could have positive or negative asymmetry. I extend Wong and Ma (2008) to show a non-monotonicity property of preference when a scale parameter changes sign. The non-monotonicity property is related to the fact that adding (deducting) a zero-conditional-mean random variable to a random outcome is disliked (preferred) by risk averse investors. I characterize indifference curves, define the three-parameter dominance and efficient set in the three-parameter space. This part of results are applicable to the three-parameter location-scale distributions. Further extension to location-scale distributions with more than three parameters is straightforward.

For the factor-recentered skew normal distribution and Simaan's three-parameter distributions, I show that the efficient set is a part of conical surface bounded by two lines. Interestingly, the mean–variance-skewness efficient set for these distributions also possesses similar geometric properties.

With constant absolute risk aversion (CARA) utility and factor-recentered skew normal asset returns, the closed-form certainty equivalent is obtained. The benefit of considering higher moments can be analytically obtained by comparing this certainty equivalent to the mean–variance certainty equivalent. Computational effort in numerical analysis can be significantly reduced using this certainty equivalent because maximizing the expected utility is equivalent to maximizing its certainty equivalent. Several numerical examples are analyzed to compare the compositions of optimal portfolio when parameters change. The first-order conditions of maximizing the certainty equivalent provide a new asset pricing model which nests the CAPM.

The rest of the paper is organized as follows. Section 2 discusses the skew normal distribution and its parameterization. Section 3 discusses the factor-recentered skew normal distribution and its factor model expression. Section 4 analyzes the non-monotonicity property of scale preference and characterize indifference curves. Section 5 characterizes the efficient set. Section 6 presents numerical analysis and an asset pricing model with CARA utility and its certainty equivalent and Section 7 concludes.

2. Skew normal distribution

Azzalini (1985) proposes and analyzes the univariate skew normal distribution. Azzalini and Dalla Valle (1996) extend the work to propose the multivariate skew normal distribution. There are several equivalent methods to construct the multivariate skew normal distribution. I focus on the conditioning method discussed by Branco and Dey (2001).

Consider $\mathbf{X}=[X_1,\dots,X_n]^\top$ a random vector. Let $[\mathbf{X}^\top, X_0]^\top$ be an (n+1)-dimensional multivariate normal random vector

$$\begin{pmatrix}\mathbf{X}\\ X_0\end{pmatrix}\sim N_{n+1}\left[\begin{pmatrix}\boldsymbol{\mu}\\ 0\end{pmatrix},\begin{pmatrix}\boldsymbol{\Omega} & \boldsymbol{\delta}\\ \boldsymbol{\delta}^\top & 1\end{pmatrix}\right]. \tag{1}$$

The random vector $\mathbf{Y}=[\mathbf{X}|X_0>0]$ follows the multivariate skew normal distribution. Bayes' theorem can be used to derive the density function of $\mathbf{Y}$:

$$f_{\mathbf{Y}}(\mathbf{y})=\frac{P(X_0>0|\mathbf{y})f_{\mathbf{X}}(\mathbf{y})}{P(X_0>0)}.$$

From the multivariate normal construction in (1), $\mathbf{X}\sim N_n(\boldsymbol{\mu},\boldsymbol{\Omega})$, $X_0|\mathbf{y}\sim N(\boldsymbol{\delta}^\top\boldsymbol{\Omega}^{-1}(\mathbf{y}-\boldsymbol{\mu}),1-\boldsymbol{\delta}^\top\boldsymbol{\Omega}^{-1}\boldsymbol{\delta})$, and $P(X_0>0)=1/2$. The multivariate skew normal density is[2]

$$f_{\mathbf{Y}}(\mathbf{y};\boldsymbol{\mu},\boldsymbol{\Omega},\boldsymbol{\alpha})=2\phi_n(\mathbf{y};\boldsymbol{\mu},\boldsymbol{\Omega})\Phi(\boldsymbol{\alpha}^\top(\mathbf{y}-\boldsymbol{\mu})), \tag{2}$$

where $\phi_n(\cdot;\boldsymbol{\mu},\boldsymbol{\Omega})$ is the n-dimensional multivariate normal p.d.f. and $\Phi(\cdot)$ is the c.d.f. of the standard normal. $\boldsymbol{\mu}$ is the location parameter and $\boldsymbol{\Omega}$ is the scale parameter. $\boldsymbol{\alpha}$ is the asymmetry parameter and has the following expression:

$$\boldsymbol{\alpha}=\frac{1}{(1-\boldsymbol{\delta}^\top\boldsymbol{\Omega}^{-1}\boldsymbol{\delta})^{1/2}}\boldsymbol{\Omega}^{-1}\boldsymbol{\delta}. \tag{3}$$

[1] Azzalini's homepage (http://azzalini.stat.unipd.it/SN/) lists about 100 published papers related to skew normal prior to year 2008.

[2] Azzalini and Dalla Valle (1996) use different construction methods and obtain the same density.