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Pricing Parisian and Par*asian* options analytically

Song-Ping Zhu *, Wen-Ting Chen

School of Mathematics and Applied Statistics, University of Wollongong, NSW 2522, Australia

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ABSTRACT

In this paper, two analytic solutions for the valuation of European-style Parisian and Par*asian* options under the Black–Scholes framework are, respectively, presented. A key feature of our solution procedure is the reduction of a three-dimensional problem to a two-dimensional problem through a coordinate transform designed to combine the two time derivatives into one. Compared with some previous analytical solutions, which still require a numerical inversion of Laplace transform, our solutions, written in terms of double integral for the case of Parisian options but multiple integrals for the case of Par*asian* options, are both of explicit form; numerical evaluation of these integrals is straightforward. Numerical examples are also provided to demonstrate the correctness of our newly derived analytical solutions from the numerical point of view, through comparing the results obtained from our solutions and those obtained from adopting other standard finite difference approaches.

1. Introduction

Parisian and Par*asian* options are simple extensions of classical barrier options, with a "trigger" device added on, mainly for the purpose of preventing option traders from deliberately manipulating the underlying asset price when it is close to the barrier, in order to gain advantage in the option position they hold. However, while the concept of counting the duration of breaching the barrier as part of the barrier condition has made it possible for them to be used in other types of financial derivatives (e.g., the convertibility and callability features in convertible bonds (Kwok and Lau, 2001) and the reorganization upon financial distress in structural models of defaultable bonds, as pointed out in Bernard et al. (2005), such a simple addition of a financial clause has caused considerable difficulties in quantitatively pricing these options, at least analytically. In this paper, two closed-form analytic formulae for the prices of Parisian and Par*asian* options are presented for the first time.

While both Parisian options and Par*asian* options share the same feature that there is a separate "clock" set up to record the total time that the underlying has passed the barrier (either above or below, depending on the type of the barrier option), the main difference between them is how the clock is reset. If one accumulates the time spent in a row and resets it to zero each time the underlying price crosses the barrier, this type is referred to as continuous Parisian options, or simply Parisian options. On the other hand, if one adds the time spent below or above the barrier without resetting the accumulated time to zero each time the underlying crosses the barrier, these options are named as cumulative Parisian options, or simply Par*asian* options. For simplicity of reference, we may sometimes in this paper refer to these two options as "Parisian-type options" when there is no need to distinguish them.

* Corresponding author. Tang-Au-Qing Chair Professor (adjunct) of Jilin University, Jilin 130012, China. Tel.: +61 2 42213807; fax: +61 2 42214845.
E-mail address: spz@uow.edu.au (S.-P. Zhu).

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The valuation problem of Parisian-type options has been recognized as much more difficult than that of classical barrier options (Kwok, 2008). While a closed-form analytical solution for the latter has already been found (Kwok, 2008), the former could only be solved approximately. The difficulty of pricing Parisian-type options mainly comes from the co-existence of two different barriers specified in the option contracts: a barrier of the underlying asset price and a barrier time, which is defined as the accumulated time that the underlying has spent above or below the barrier.

In the literature, predominately two types of valuation techniques, the quasi-analytic approaches and the numerical methods, are well documented for the pricing of Parisian-type options. Of all the quasi-analytic methods, the most influential approach was the one proposed by Chesney et al. (1997a,b). They used the theory of Brownian excursions and defined the value of a Parisian option in terms of an integral expressed as an inverse Laplace transform. Similar works with slightly different forms of the Laplace transform were carried out later by Hugonnier (1999), and Schröder (2003) to price various Parisian-type options. One of the fatal drawbacks of all these earlier papers is that the closed-form solutions found are all in the Laplace space only; an inverse Laplace transform needs to be performed before these solutions can be actually adopted. However, it is well known that numerically performing Laplace inversion is an inverse problem, which is usually unstable and sensitive to round-off errors (Kwok and Barthez, 1989; Cheng et al., 1994). Labart and Lelong (2009) did develop a fast and stable numerical method to invert the prices of Parisian options, but the extension of their method to the Par*asian* case is not discussed at all. Bernard et al. (2005) also developed a simple method to compute the inverse Laplace transform associated with the pricing of Parisian-type options. But, they used functions which admit known Laplace transform inversions to approximate the option price in the Laplace space first and then invert them analytically. This approach has virtually resulted in an approximate solution rather than a closed-form exact solution as we shall present in this paper. Numerical methods, as another good alternative, were also intensively developed in the past. A typical method in this category is the PDE (Partial Differential Equation) approach presented in Haber et al. (1998), in which two PDE systems governing the prices of Parisian and Par*asian* options are established and then solved using the explicit finite difference scheme. Whilst flexible and easy to implement, there are some deficiencies of their approach, as shall be pointed out later.

In this paper, we present two closed-form analytic solutions for the valuation of Parisian and Par*asian* options, respectively, under the BS (Black–Scholes) framework. Based on several reasonable financial arguments, two new PDE systems for the prices of Parisian and Par*asian* options have been established first, with one boundary condition added on for each system to ensure its closeness. The newly established PDE systems are then simplified through a coordinate transform that has elegantly "absorbed" one dimension associated with the "barrier time" into the time direction. The purely analytical procedures, adopted afterwards, differ w.r.t. (with respect to) the resetting mechanisms specified in the option contracts. For a Parisian option, the resulted simplified PDE system is solved analytically by applying the Laplace transform technique together with the construction of "moving windows" to evaluate the option prices backwards, slide by slide, until the given time has been reached, whereas for a Par*asian* option, its non-resetting mechanism has obstructed the application of the "moving window" technique, and we apply the double Laplace transform as an alternative to analytically solve for its option price. Finally, through Laplace inversions, two completely analytic closed-form solutions are obtained for the prices of Parisian and Par*asian* options, respectively. It should be pointed out that our explicit pricing formulae for pricing Parisian-type options should be valuable in both theoretic and practical senses. Theoretically, although there are several existing methods to price Parisian-type options (e.g., those with analytical solutions found in the Laplace space but still require a numerical Laplace inversion to obtain the option price in the original time domain, as presented in Bernard et al. (2005), Chesney et al. (1997a,b), Hugonnier (1999), Labart and Lelong (2009), and Schröder (2003)), explicit and closed-form exact solutions are presented for the first time.[1] Practically, the final form of our solutions, written in terms of a linear combination of several integrals, can be used to price Parisian-type options accurately and efficiently. With a growing demand of trading exotic options in today's finance industry, our solution procedures may lead to the development of pricing formulae for other exotic derivatives, such as the Edokko options introduced in Fujita and Miura (2003), which are generalizations of both Parisian and Delayed Barrier options.

2. PDE systems for pricing Parisian and Par*asian* options

As pointed out previously, under the BS framework, the PDE systems for the prices of Parisian-type options have already been established in Haber et al. (1998). However, the complexities associated with their PDE systems have hindered the application of various analytic methods. In this section, two simplified PDE systems governing the prices of Parisian-type options are provided, which pave the way for the achievement of closed-form analytic solutions for both options.

2.1. Parisian options

A Parisian option is a special kind of barrier options for which the knock-in or knock-out feature is only activated if the underlying remains continually in breach of the barrier $\overline{S}$ for a pre-specified time period $\overline{J}$. Like classical barrier options,

[1] A solution written in terms of the inverse Laplace transform without the inversion being carried out analytically is still of closed form. However, since numerical inversion of Laplace transform is an ill-posed problem, this kind of solutions is not truly "explicit" as far as the computation of the numerical values of an option is concerned.