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The method of finite spheres in three-dimensional linear static analysis

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ABSTRACT

The objective of this paper is to assess the reliability and effectiveness of the method of finite spheres, a truly meshless overlapping finite element method, for the solution of practical three-dimensional linear elasticity problems. Advantages include simplified discretization and the elimination of element distortion. The method is implemented in the ADINA finite element program through a user-supplied element subroutine. The solutions of three increasingly complex three-dimensional problems are studied (1) to establish the reliability of the method for practical linear elasticity problems and (2) to assess the effectiveness of the method as compared to the standard finite element method. The solutions indicate that the method of finite spheres is between one and two orders of magnitude more expensive in computational time than the standard finite element method. This is still a promising result since there are significant time savings for the method of finite spheres during the pre-processing phase, particularly in the discretization of complicated three-dimensional geometries and because the overlapping sphere elements can be directly coupled to traditional finite elements.

1. Introduction

The method of finite spheres is a truly meshless overlapping finite element method developed to overcome the challenges in mesh-based numerical methods. In the finite element method, challenges include mesh generation for geometrically complex domains, avoidance of severe element distortions, and mesh alignment and refinement for modeling problems with discontinuities and singularities [1]. Mesh generation is time-consuming and requires special attention to remove distorted elements, especially for complex three-dimensional domains. Element distortion causes a loss of predictive capability since the element is no longer able to represent the same order of polynomials, leading to inaccuracies in numerical integration and an overall loss of reliability and solution accuracy [2]. In the method of finite spheres, overlapping sphere elements simplify the discretization of complex three-dimensional domains and eliminate the risk of distorted elements.

The challenges in mesh-based numerical methods have attracted substantial research efforts, leading to the development of numerous meshless methods [3–5]. Some of the most prominent meshless methods include smoothed particle hydrodynamics (SPH), the diffuse element method (DEM), the element-free Galerkin method (EFG), and the meshless local Petrov–Galerkin method (MLPG). The method of finite spheres (MFS) inherently possesses the advantages of meshless methods, and can also be thought of as a reliable and efficient finite element method using overlapping elements.

Smoothed particle hydrodynamics, one of the earliest developments in meshless methods, was originally used to model astrophysical phenomena. The method has since been implemented for a wide range of practical engineering applications, predominantly in the area of computational fluid dynamics, but also with extensions to solid mechanics. Despite the inherent advantages of being a meshless Lagrangian particle method, SPH possesses some numerical complications such as tensile instability and spurious boundary effects, which can lead to poor accuracy in the solution. Furthermore, generally a large number of particles and the use of adjustable solution factors are required to obtain reasonable accuracy, reducing the efficiency and robustness of the method. Several modifications and corrections have been proposed to restore consistency and accuracy of SPH, but further research efforts are necessary before the method can be regarded as robust and efficient for practical applications [6,7].

The diffuse element method was the first of many meshless methods based on the Galerkin formulation. The method uses moving least squares (MLS) to generate smooth approximations based on a set of discretization points. Since DEM is a global weak form method, a background mesh is required for numerical integration, suggesting the method is only meshless with regard to constructing interpolation functions. Furthermore, there are a number of oversimplifications which affect the validity of the

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method. In particular, the derivative of the approximation functions is evaluated only approximately, a very low quadrature rule for numerical integration is applied, and the Dirichlet boundary conditions are not accurately enforced. Consequently, DEM does not pass the patch test and fails consistency requirements [8].

The element-free Galerkin method is an extension of DEM, introducing a series of improvements which results in a more accurate formulation at the expense of increased computational cost. Specifically, EFG correctly evaluates the derivatives of the approximation functions, employs a larger number of integration points in the numerical integration procedure, and utilizes Lagrange multipliers to accurately enforce the Dirichlet boundary conditions. The EFG method also uses MLS approximations to construct the trial and test functions which provides reasonable accuracy, but requires an expensive matrix inversion at every integration point. Furthermore, there is an additional condition that at every integration point there is a minimum number of domains of influence that must have nonzero support. These complications concerning matrix inversion and overlap significantly reduce the computational efficiency of the method [9–12].

The meshless local Petrov–Galerkin method is a concept that can adopt trial and test functions from different approximation spaces, resulting in various formulations which offer flexibility to deal with different boundary value problems. Various formulations of the MLPG approach have been used to solve three-dimensional elastostatics problems, using different test functions, such as the Heaviside function or the Dirac delta function, and different approximations, based on radial basis functions or moving least squares. Unlike DEM and EFG, the MLPG method works with a local weak form instead of a global weak form, which means that numerical integration is performed over local subdomains rather than using a background mesh or cell structure. Therefore, it is a truly meshless method since a mesh is not required for either interpolation or integration. However, with the approximation functions based on the MLS approximation, the method suffers from the same complications as DEM and EFG [13–15].

Although a variety of meshless techniques have been developed, the currently available reliable methods are much more expensive than the finite element method and come with various complications that affect their overall effectiveness. The method of finite spheres incorporates advantages of the finite element method and meshless methods and focuses on being both reliable and computationally efficient. Early research demonstrated the reliability of MFS for one- and two-dimensional linear analysis of solids and fluids. Further research established a mixed displacement/pressure formulation, improved numerical integration, finite element coupling, enrichment strategies, automatic discretization, genetic algorithms for numerical integration, and a scheme for the analysis of wave propagation problems [16–25].

The focus of this paper is on assessing the reliability and efficiency of the method of finite spheres for the analysis of practical three-dimensional linear elastic problems, where the traditional finite element method suffers from costly mesh generation and errors resulting from element distortions. In Section 2, we develop the theory and formulation of the method of finite spheres and present an effective local approximation space for constructing three-dimensional interpolation functions. Thereafter, in Section 3, we propose a simple numerical integration scheme known as the piecewise Gauss–Legendre quadrature rule for the integration of the nonpolynomial functions over the three-dimensional spherical domains. In Section 4, we discuss our implementation of the method of finite spheres in a user element subroutine of ADINA. Then in Section 5, we study the solutions of three increasingly more complex three-dimensional analysis problems in order to establish the reliability and assess the efficiency of MFS for practical linear elastic analysis. Lastly, in Section 6, we summarize the major developments and discuss possible further research toward improving the efficiency of the method.

2. Formulation of the method of finite spheres

In this section we present the theory and formulation of the method of finite spheres for three-dimensional linear elasticity problems. The presentation is largely based on Ref. [16].

2.1. Sphere discretization

Consider a general three-dimensional domain V with domain boundary $S = S_u \cup S_f$, where S_u is the Dirichlet boundary and S_f is the Neumann boundary. The unit normal to the domain boundary, $\underline{n}$, is positive in the outward direction. Let $\{B(\underline{x}_I, r_I);\ I = 1, \ldots, N\}$ be a set of spheres, where $\underline{x}_I$ and r_I refer to the center coordinates and radius of sphere B_I, respectively, and where I is the nodal label of each sphere and N is the total number of spheres. As illustrated in Fig. 1, spheres can be classified as either an interior or boundary sphere.

The requirements for a valid sphere discretization are (1) all sphere centers must be within the domain, (2) the domain must be completely covered by the union of all spheres, and (3) no sphere can be completely included in any other sphere. Discretization depends only on the position vector and radius of the spheres. With overlapping elements, the method of finite spheres avoids discretization difficulties and element distortion.

2.2. Interpolation scheme

The interpolation scheme for the method of finite spheres is based on the partition of unity paradigm [26–28]. Interpolation functions are defined as the product of Shepard functions and local basis functions. An effective local approximation space is chosen for three-dimensional linear elasticity problems.

2.2.1. Shepard partition of unity functions

The Shepard partition of unity functions are given by

$$\varphi_I^0(\underline{x}) = \frac{W_I}{\sum_{J=1}^{N} W_J}, \quad I = 1, \ldots, N \tag{1}$$

where $W_I(\underline{x})$ denotes a positive radial weighting function. The Shepard functions are nonpolynomial and have zeroth-order consistency, ensuring that rigid body modes can be reproduced exactly. The choice of weighting function should consider the continuity class and the ease of differentiation and integration so that low-cost partitions of unity are obtained. We choose the quartic spline weighting function defined as

$$W_I(s) = \begin{cases} 1 - 6s^2 + 8s^3 - 3s^4, & 0 \leqslant s \leqslant 1 \\ 0, & s > 1 \end{cases} \tag{2}$$

where $s = (\|\underline{x} - \underline{x}_I\|)/r_I$.

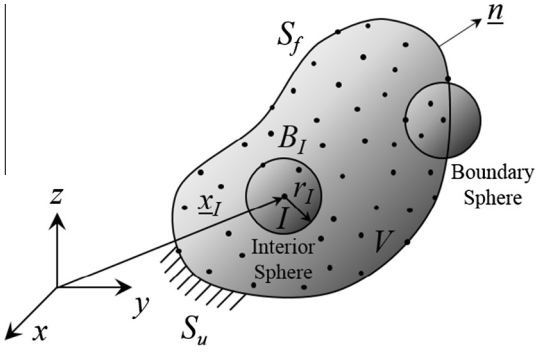

Fig. 1. General three-dimensional domain V with domain boundary $S = S_u \cup S_f$.