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Global stochastic properties of dynamic models and their linear approximations

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ABSTRACT

The dynamic properties of micro based stochastic macro models are often analyzed through a linearization around the associated deterministic steady state. Recent literature has investigated the errors made by such a deterministic approximation. Complementary to this literature we investigate how the linearization affects the stochastic properties of the original model. We consider a simple real business cycle model with noisy learning by doing. The solution has a stationary distribution that exhibits moment failure and has an unbounded support. The linear approximation, however, yields a stationary distribution with possibly a bounded support and all moments finite.

1. Introduction

The dynamic properties of micro based stochastic macro models are often analyzed through a linearization around the associated deterministic steady state. In the seminal paper on real business cycles (RBC) Kydland and Prescott (1982) employed first order approximations to solve their dynamic, stochastic general equilibrium (DSGE) model. This method has become highly popular in analyzing DSGEs. Campbell (1994) and Uhlig (1997) provide overviews on how to perform the linearization of the dynamic micro based stochastic macro models. A number of papers has investigated the accuracy of the log linear approximation, by looking at the deterministic part of the approximate solution. Tesar (1995) and Kim (1997) prove that the loglinear approximation method may create welfare reversals, to the extent that the incomplete-markets economy produces a higher level of welfare than the complete-markets economy. Jin and Judd (2002) therefore recommend the use of second order perturbation methods. Sutherland (2002) and Kim and Kim (2003) have developed a bias selection method which can be as accurate as the perturbation method, but which requires less computational effort. The performance of the linear approximation in stochastic neoclassical growth models is studied by Dotsey and Mao (1992), and more recently in Arouba et al. (2006) and Fernandez-Villaverde and Rubio-Ramirez (2005).

We contribute to this literature by showing how the stochastic properties of the approximate solution differ from the equilibrium of the nonlinear model. In particular, we investigate the simplest model in the business cycle literature with

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fixed labor supply, total depreciation of capital and a log-utility function. To this we add noisy learning by doing. The solution of the resulting stochastic difference equation has a stationary distribution which exhibits moment failure and has an unbounded support. The first order approximation, however, yields a stationary distribution with bounded support and all moments finite. Thus the linear approximation dramatically alters the stochastic properties of the model. We also consider briefly an application from asset pricing with stochastic volatility.

This note is organized as follows. In Section 2 we analyze the RBC model and we show that while the exact solution of the model for the log of capital follows a stationary distribution with unbounded support and exhibits moment failure, the approximation may nevertheless have bounded support and all moments finite. Section 3 further discusses the effects of linearization in the capital asset pricing model with changing conditional volatility of the ARCH variety. Section 4 concludes.

2. Application on the real business cycle model

Log-linearization is a well known method for solving business cycle models. It has its pros and cons, which are usually discussed in a deterministic setting. We join this literature by showing how linearization may change the stochastic equilibrium behavior of the solution of a dynamic RBC model.

The environment of the basic RBC model with fixed unitary labor supply and noisy learning by doing is as follows:

1. The production function is Cobb–Douglas $Y_t = I_t^{\alpha} K_t^{1-\alpha}$, where I is technology and K is capital.
2. With full depreciation, the next period capital equals the current period's savings: $K_{t+1} = I_t^{\alpha} K_t^{1-\alpha} - C_t$.
3. The representative agent expected utility is: $U = E_t\left[\sum_{i=0}^{\infty} \beta^i \log(C_{t+i})\right]$.
4. Technological progress stems from learning by doing: $I_{t+1} = \phi_{t+1} Y_t^{\varepsilon_{t+1}}$, where $\phi_t > 0$ and ε_t are random variables independently distributed with mean ϕ and ε, respectively. The learning by doing effect stems from the aggregate production level. This externality is not taken into account by the individual consumer when planning his consumption pattern.
5. The gross rate of return on a one period investment in capital R_{t+1} equals the marginal product of capital: $R_{t+1} = (1-\alpha)(I_{t+1}/K_{t+1})^{\alpha}$.

This special case of a stochastic dynamic general equilibrium model with full depreciation of capital and log utility function admits an exact solution. The first order condition for utility maximization is: $1/C_t = \beta E_t[(1-\alpha) I_{t+1}^{\alpha} K_{t+1}^{-\alpha} / C_{t+1}]$. In order to solve the system

$$\frac{1}{C_t} = \beta E_t\left[\frac{(1-\alpha) I_{t+1}^{\alpha} K_{t+1}^{-\alpha}}{C_{t+1}}\right] \tag{1}$$

$$K_{t+1} = I_t^{\alpha} K_t^{1-\alpha} - C_t \tag{2}$$

$$Y_t = I_t^{\alpha} K_t^{1-\alpha} \tag{3}$$

$$I_{t+1} = \phi_{t+1} Y_t^{\varepsilon_{t+1}} \tag{4}$$

we guess the policy function

$$C_t = \mu I_t^{\alpha} K_t^{1-\alpha} \tag{5}$$

Inserting (5) in (1) and using the equation for the capital accumulation process (2) determines the constant $\mu = 1 - \beta(1-\alpha)$.

Subsequently substitute (5) and (3) into (2). This shows that the log of capital k_{t+1} satisfies[1]

$$k_{t+1} = \log\beta(1-\alpha) + y_t \tag{6}$$

Transform (4) into logs

$$i_{t+1} = \log\phi_{t+1} + \varepsilon_{t+1} y_t \tag{7}$$

Advancing (3) one period, taking logarithms as well and inserting (6) and (7), we obtain the first order stochastic difference equation for log income:

$$y_{t+1} = (1-\alpha)\log\beta(1-\alpha) + \alpha\log\phi_{t+1} + (\alpha\varepsilon_{t+1} + 1 - \alpha) y_t \tag{8}$$

This difference equation can be conveniently summarized as

$$X_t = A_t + B_t X_{t-1}, \tag{9}$$

[1] In this section capital letters stand for level values and small letters for log transformed variables.