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Dynamic hedging of synthetic CDO tranches with spread risk and default contagion

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A B S T R A C T

The paper is concerned with the hedging of credit derivatives, in particular synthetic CDO tranches, in a dynamic portfolio credit risk model with spread risk and default contagion. The model is constructed and studied via Markov-chain techniques. We discuss the immunization of a CDO tranche against spread- and event risk in the Markov-chain model and compare the results with market-standard hedge ratios obtained in a Gauss copula model. In the main part of the paper we derive model-based dynamic hedging strategies and study their properties in numerical experiments.

1. Introduction

The risk management for books of synthetic CDO tranches has become an issue of high concern for many investors on credit markets. Typically an investor has taken a protection-seller position in one or several CDO tranches and tries to offset the ensuing risk by taking an opposite (protection-buyer) position in the single-name credit default swaps (CDSs) or in the CDS-index underlying the tranche. In practice, the size of the hedging positions is determined by a pragmatic approach, akin to the use of duration in interest rate risk management: in order to protect a CDO tranche against fluctuations in credit spreads the tranche is first priced via the Gauss copula model, using observed CDS spreads and implied-correlation methodology to determine the model parameters. Next one varies the swap spread of one of the underlying names, name k, say, and defines the so-called *spread delta* of that name as the ratio of the change in the market value of the CDO tranche and of a CDS on name k. The hedge ratios immunizing the tranche against a change in the index spread are determined in a similar way. Sometimes investors also seek to protect their position against defaults in the underlying reference pool (hedging of jump-to-default risk). The hedge ratio immunizing a CDO tranche against the default of firm k is known as *jump-to-default ratio*; it is computed as the ratio of the loss due to the default of that name in the tranche and in the CDS on name k. In computing these losses it is assumed that the credit spreads of the surviving firms are not affected by the default event. Further details on market-standard hedging practice can for instance be found in Neugebauer (2006).

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The market-standard approach has a number of problems: contagion effects (the fact that credit spreads of surviving firms often jump in reaction to default events) are neglected, and there is no theoretically consistent methodology supporting the definition of spread deltas and jump-to-default ratios. These issues are clearly important also from a practical point of view. To begin with, the current financial crisis underlines the relevance of default contagion on credit markets—just think of the events surrounding the default of Lehman Brothers—and neglecting contagion effects may lead to inappropriate hedge ratios. Moreover, it is well-known from markets for other types of derivatives that ad hoc hedging strategies frequently lead to unaccounted drift- and time-decay effects (see for instance El Karoui et al., 1998). The lack of a sound hedging methodology for portfolio credit derivatives is of course closely related to the fact that the market-standard copula models are static, so that theoretically consistent dynamic hedging strategies cannot be derived in copula models. Note that this deficiency is inherent in the copula framework; it cannot be rectified by using more sophisticated copulas than the Gauss copula.

In this paper we make an attempt to address these issues. In Section 2, we propose a dynamic credit risk model which allows for the explicit modelling of default contagion and spread risk, and which is therefore an ideal workbench for analyzing the hedging of CDO tranches. The model belongs to the class of models with interacting default intensities such as Jarrow and Yu (2001), Davis and Lo (2001), Giesecke and Weber (2006), Bielecki et al. (2008), or Herbertsson (2008); it is closely related to the Markov-chain models within the so-called top-down approach to credit portfolio modelling studied by Arnsdorf and Halperin (2007), Lopatin and Misirpashaev (2007) or Cont and Minca (2008). In Section 3 we give a formal description of the cash-flow dynamics of CDSs and CDOs. In Section 4 we compute jump-to-default ratios and spread deltas for the Markov-chain model and compare the results with the market-standard values from a Gauss copula model. It turns out that in many cases the hedge ratios differ substantially, mainly because of contagion effects. In Section 5 we study the dynamic replication of CDO tranches using martingale representation results for marked point processes. We find that in the special case where credit spreads evolve deterministically between default times (pure jump-to-default risk) the market is complete; the dynamic replication strategy coincides with the jump-to-default ratio for the Markov-chain model. With spread risk and jump-to-default risk on the other hand markets are typically incomplete, so that we resort to the concept of risk-minimization introduced by Föllmer and Sondermann (1986). Numerical experiments further illustrate certain properties of risk-minimizing hedging strategies. It is shown that risk-minimizing hedging strategies interpolate between the hedging of spread- and jump-to-default risk and that deviations from the popular assumption of a homogeneous portfolio can have a sizeable impact on the form and on the performance of hedging strategies. At this point it is worth mentioning that while in the present paper we concentrate on CDO tranches, the theoretical results we obtain apply to many other credit derivatives such as index spread options or basket swaps, often with merely notational changes.

The dynamic hedging of credit risky securities is studied among others by Bielecki et al. (2004), Elouerkhaoui (2006), Bielecki et al. (2007) and Laurent et al. (2008). The latter two papers are closely related to our contribution. Laurent et al. (2008) study the hedging of CDO tranches via dynamic trading in CDS indices in the Markov-chain model of Frey and Backhaus (2008). However, they concentrate on the case without spread risk and hence on complete markets. Bielecki et al. (2007) derive interesting theoretical results on the hedging of basket swaps in a generic dynamic credit portfolio model without spread risk. Since the hedging against random fluctuations of credit spreads is an issue of high concern for practitioners, we believe that the inclusion of spread risk and the application of incomplete-market methodology is an important extension over these papers. Rosen and Saunders (2009) derive interesting results on *static* hedging strategies for CDOs.

2. The model

2.1. Notation

Fix some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), Q)$. The $\sigma-$field $\mathcal{F}_t$ represents the information available to investors at time t; all processes introduced below will be $(\mathcal{F}_t)-$adapted. We consider a fixed portfolio of m firms, indexed by $i \in \{1, \ldots, m\}$. The $(\mathcal{F}_t)-$stopping time τ_i with values in $(0, \infty)$ represents the default time of firm i. The default state of the portfolio is thus described by the *default indicator process* $Y = (Y_{t,1}, \ldots, Y_{t,m})_{t \geq 0}$ where $Y_{t,i} = 1_{\{\tau_i \leq t\}}$; note that $Y_t \in \{0,1\}^m$. For simplicity we assume that the exposure of each firm is normalized to one. Denoting the percentage loss given default (LGD) of firm i at time t by the predictable random variable $\delta_{t,i} \in (0,1]$, the *loss state* $L_t = (L_{t,1}, \ldots, L_{t,m})$ of the portfolio and the *aggregate portfolio loss* $\overline{L}_t$ at time t are given by

$$L_{t,i} = \int_0^t \delta_{s,i}\, dY_{s,i}, \quad 1 \leq i \leq m \quad \text{and} \quad \overline{L}_t = \sum_{i=1}^m L_{t,i}. \tag{1}$$

Moreover, Y_t can be recovered from L_t via $Y_{t,i} = 1_{\{L_{t,i} > 0\}}$, $1 \leq i \leq m$. Since we consider only models without simultaneous defaults, we can define the ordered default times $T_0 < T_1 < \cdots < T_m$ recursively by

$$T_0 = 0 \quad \text{and} \quad T_n = \min\{\tau_i : 1 \leq i \leq m, \tau_i > T_{n-1}\}, \quad 1 \leq n \leq m.$$