Journal of Economic Dynamics & Control 33 (2009) 283–295

Contents lists available at ScienceDirect

Journal of Economic Dynamics & Control

journal homepage: www.elsevier.com/locate/jedc

Computing the mean square error of unobserved components extracted by misspecified time series models

Andrew C. Harvey *, Davide Delle Monache

Faculty of Economics, University of Cambridge, Sidgwick Avenue, Cambridge CB3 9DD, UK

ARTICLE INFO

Article history:
Received 19 February 2008
Accepted 31 May 2008
Available online 18 June 2008

JEL classification:
C32

Keywords:
Detrending
Exponentially weighted moving average
Hodrick–Prescott filter
Kalman filter
Smoother

ABSTRACT

Algorithms are presented for computing mean square errors in a misspecified unobserved components model when the true model is known. It is assumed that both the true and misspecified models can be put in linear state space form. The algorithm for filtering is based on the Kalman filter while that for smoothing modifies the fixed-point smoother. Illustrations include the efficiency of the Hodrick–Prescott filter for annual flow data and the mean square error of predictions for misspecified models from the autoregressive integrated moving average class.

1. Introduction

This article presents algorithms for computing the mean square errors (MSEs) of estimators of components in an unobserved components (UC) time series model obtained by a linear filter constructed so as to be optimal for a misspecified model. It is assumed that both the true and misspecified models can be put in state space form (SSF). The algorithm for filtering is based on the Kalman filter (KF) while that for smoothing is derived from the fixed-point smoother. Because many seemingly *ad hoc* filters, such as the exponentially weighted moving average (EWMA) and the detrending method of Hodrick and Prescott (1997), can be rationalized by a UC model, the algorithms can be applied in a wide range of situations. The results provide insight into the effects of misspecification and have implications for robustness.

The general linear SSF applies to a multivariate time series, $\mathbf{y}_t$, containing N elements. These observable variables are related to an $m \times 1$ state vector, $\boldsymbol{\alpha}_t$ through the following measurement equation:

$$\mathbf{y}_t = \mathbf{Z}_t \boldsymbol{\alpha}_t + \boldsymbol{\varepsilon}_t, \quad t = 1, \ldots, T, \tag{1}$$

where $\mathbf{Z}_t$ is an $N \times m$ matrix and $\boldsymbol{\varepsilon}_t$ is an $N \times 1$ vector of serially uncorrelated disturbances with mean zero and covariance matrix $\mathbf{H}_t$, that is $\mathrm{E}(\boldsymbol{\varepsilon}_t) = \mathbf{0}$ and $Var(\boldsymbol{\varepsilon}_t) = \mathbf{H}_t$. The state vector is generated by the following transition equation:

$$\boldsymbol{\alpha}_t = \mathbf{T}_t \boldsymbol{\alpha}_{t-1} + \boldsymbol{\eta}_t, \quad t = 1, \ldots, T, \tag{2}$$

where $\mathbf{T}_t$ is an $m \times m$ matrix and $\boldsymbol{\eta}_t$ is a $m \times 1$ vector of serially uncorrelated disturbances with mean zero and covariance matrix, $\mathbf{Q}_t$, that is $\mathrm{E}(\boldsymbol{\eta}_t) = \mathbf{0}$ and $Var(\boldsymbol{\eta}_t) = \mathbf{Q}_t$.

* Corresponding author. Tel.: +44 1223 335228; fax: +44 1223 335475.
E-mail address: andrew.harvey@econ.cam.ac.uk (A.C. Harvey).

0165-1889/$ - see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.jedc.2008.05.007

The specification of the state space system is completed by assuming that the initial state vector, $\boldsymbol{\alpha}_0$, has a mean of $\mathbf{a}_0$ and a covariance matrix $\mathbf{P}_0$, that is $\mathrm{E}(\boldsymbol{\alpha}_0) = \mathbf{a}_0$ and $Var(\boldsymbol{\alpha}_0) = \mathbf{P}_0$, where $\mathbf{P}_0$ is positive semi-definite (PSD), and that the disturbances $\boldsymbol{\varepsilon}_t$ and $\boldsymbol{\eta}_t$ are uncorrelated with the initial state, that is $\mathrm{E}(\boldsymbol{\varepsilon}_t\boldsymbol{\alpha}_0') = \mathbf{0}$ and $\mathrm{E}(\boldsymbol{\eta}_t\boldsymbol{\alpha}_0') = \mathbf{0}$ for $t = 1, \ldots, T$. It will be assumed that the disturbances are uncorrelated with each other in all time periods, that is $\mathrm{E}(\boldsymbol{\varepsilon}_t\boldsymbol{\eta}_s') = \mathbf{0}$ for all $s, t = 1, \ldots, T$, though this assumption may be relaxed, the consequence being a slight complication in some of the filtering formulae; see Appendix C.

The system matrices $\mathbf{Z}_t, \mathbf{T}_t$, $\mathbf{H}_t$ and $\mathbf{Q}_t$, $t = 1, \ldots, T$, are non-stochastic and so the system is linear. Hence the KF gives $\mathbf{a}_t$, the minimum mean square error linear estimator (MMSLE) of $\boldsymbol{\alpha}_t$ based on information up to time t, together with its MSE matrix $\mathbf{P}_t$. The smoother gives the MMSLE of $\boldsymbol{\alpha}_t$ based on all the information up to time T.

The algorithms presented below use knowledge of the true model to find the MSE of an estimator of the state vector constructed using the filter from a misspecified model. The SSF of the misspecified model has system matrices $\mathbf{T}_t$, $\mathbf{Z}_t$, $\widehat{\mathbf{Q}}_t$ and $\widehat{\mathbf{H}}_t$. In the true model, $\widehat{\mathbf{Q}}_t$ and $\widehat{\mathbf{H}}_t$ are replaced by $\mathbf{Q}_t$ and $\mathbf{H}_t$. Assuming that $\mathbf{T}_t$ and $\mathbf{Z}_t$ are the same, is not restrictive, since redundant components can be included in the state vector by treating them as fixed and equal to 0.

Section 2 presents the algorithm for filtering and shows how it may be extended to evaluate the MSE of forecasts. The smoothing algorithm is derived in Section 3. Section 4 applies the algorithms to study the efficiency and sensitivity of some commonly used filters. We first investigate the sensitivity of the EWMA and the Hodrick–Prescott (HP) filter to different values of the smoothing constant. The analysis is based on the fact that these two filters can be optimal for trend plus noise UC models where the trends are a random walk and an integrated random walk, respectively. We then explore the consequences of using an exponential smoother, firstly when the true model is an integrated random walk plus noise (IRWN), and, secondly, when the true model is a random walk plus noise (RWN) but the relative variances change half way through the sample. In Section 5 we calculate the efficiency of the HP filter when the data are assumed to be generated by a trend–cycle model of the type that may be routinely fitted with the STAMP package of Koopman et al. (2007). An interesting feature of the analysis is the contrast between the efficiency of the smoother in the middle of the sample and the efficiency of the filter at the end.

Section 6 extends the algorithms to deal with situations where the observation interval is bigger than the model interval. For example, a quarterly model may be assumed, but observations are only available annually. Handling flow variables is more difficult than handling stocks, but flows are relevant to the issue of what smoothing constant to use for the HP filter when detrending annual gross domestic product (GDP). There is a considerable body of literature on this matter; see Ravn and Uhlig (2002) and the references therein.

Finally, Section 7 illustrates how the methods can be applied when the true and/or misspecified model is from autoregressive integrated moving average class.

The computations reported were carried out using programs written in the Ox language of Doornik (1999). It is envisaged that subroutines will become incorporated in the SsfPack set of programs of Koopman et al. (1999).

2. Filtering

The misspecified filter is just the KF applied to the misspecified SSF. It yields $\widehat{\mathbf{a}}_t$, the estimator of the state based on information at time t and its MSE matrix, $\widehat{\mathbf{P}}_t$. Below we derive a parallel recursion that computes the true MSE matrix, $\mathbf{P}_t$. When the model is correctly specified this recursion reduces to the MSE recursion for the standard KF.

2.1. Form and derivation of the true MSE filter

The prediction step in the misspecified KF is

$$\widehat{\mathbf{a}}_{t|t-1} = \mathbf{T}_t\widehat{\mathbf{a}}_{t-1}, \;\; \widehat{\mathbf{y}}_{t|t-1} = \mathbf{Z}_t\widehat{\mathbf{a}}_{t|t-1}, \quad t = 1, \ldots, T,$$

while the updating step is

$$\widehat{\mathbf{a}}_t = \widehat{\mathbf{a}}_{t|t-1} + \widehat{\mathbf{P}}_{t|t-1}\mathbf{Z}_t'\widehat{\mathbf{F}}_t^{-1}\widehat{\boldsymbol{v}}_t, \quad t = 1, \ldots, T,$$

where $\widehat{\boldsymbol{v}}_t = \mathbf{y}_t - \mathbf{Z}_t\widehat{\mathbf{a}}_{t|t-1}$ is the vector of (pseudo) innovations with MSE matrix $\widehat{\mathbf{F}}_t = \mathbf{Z}_t\widehat{\mathbf{P}}_{t|t-1}\mathbf{Z}_t' + \widehat{\mathbf{H}}_t$. The corresponding steps for the MSE matrix are

$$\widehat{\mathbf{P}}_{t|t-1} = \mathbf{T}_t\widehat{\mathbf{P}}_{t-1}\mathbf{T}_t' + \widehat{\mathbf{Q}}_t$$

and

$$\widehat{\mathbf{P}}_t = \widehat{\mathbf{P}}_{t|t-1} - \widehat{\mathbf{P}}_{t|t-1}\mathbf{Z}_t'\widehat{\mathbf{F}}_t^{-1}\mathbf{Z}_t\widehat{\mathbf{P}}_{t|t-1}.$$

It is assumed that $\widehat{\mathbf{F}}_t$ is PD, for all $t = 1, \ldots, T$. The filter is initialized with $\widehat{\mathbf{a}}_0$ and $\widehat{\mathbf{P}}_0$. The updating and prediction steps can be combined in a single recursion for the *contemporaneous filter*, $\widehat{\mathbf{a}}_t$, or the *predictive filter*, $\widehat{\mathbf{a}}_{t+1|t}$.

A recursion for computing the true MSE matrix is derived as follows. Since the error at the prediction step is

$$\widehat{\mathbf{a}}_{t|t-1} - \boldsymbol{\alpha}_t = \mathbf{T}_t(\widehat{\mathbf{a}}_{t-1} - \boldsymbol{\alpha}_{t-1}) + \boldsymbol{\eta}_t, \tag{3}$$