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Cake eating, exhaustible resource extraction, life-cycle saving, and non-atomic games: Existence theorems for a class of optimal allocation problems

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ABSTRACT

This paper investigates the problem concerning the existence of a solution to a diverse class of optimal allocation problems which include models of cake eating, exhaustible resource extraction, life-cycle saving, and non-atomic games. A new formulation that encompasses all these diverse models is provided. Examples of these models for which a solution does not exist and the causes of the non-existence are studied. Two theorems are provided to tackle the existence problem under different conditions. Several analytical examples with a closed-form solution are offered to illustrate the usefulness of the existence theorems.

1. Introduction

This paper studies the problem concerning the existence of a solution to a diverse class of optimal allocation models which have appeared in various forms and found a lot of applications in economics. The optimal allocation models have the following common underlying structure:

$$\max_{c(t)\in\Phi} \int_0^1 \alpha(t)f(t)g(c(t))\,dt \tag{1}$$

subject to

$$c(t)\geqslant 0, \tag{2}$$

$$S(t)\geqslant 0, \tag{3}$$

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$$S'(t) = j(t)S(t) + m(t) - f(t)c(t), \tag{4}$$

and

$$S(0) = S_0, \tag{5}$$

where $c, S, \alpha, j, m, f : [0,1] \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$, $S_0 \in [0, \infty)$, Φ denotes the space of piecewise continuous functions, and $\mathbb{R}$ denotes the real line.

Model (1)–(5) is an optimal control problem with a control constraint and a state constraint. For convenience, the range of t is taken to be [0, 1], but it can be any bounded subset of the non-negative real line. A wide variety of economic models can be formulated in the form of (1)–(5), e.g., exhaustible resource extraction (Hotelling, 1931), cake-eating (Gale, 1967), life-cycle saving (Yaari, 1965), and non-atomic games (Aumann and Shapley, 1974). The formulation (1)–(5) provides a new and convenient way to encompass all these diverse models.

Karlin (1959, pp. 210–214) was the first to study the existence problem for a special case of (1)–(5) in which $j(t) = 0$, $m(t) = 0$, and $f(t) = 1$ are assumed. In this case, the optimization problem (1)–(5) becomes a calculus of variations problem. Yaari (1964) advances Karlin's (1959) analysis and provides an interesting example to show that the variational problem may not have a solution even when $g(\cdot)$ is strictly concave and the other functions are smooth and well defined. Yaari's (1964) example is counter-intuitive because it means that there is no optimal way to allocate a given amount of resources to maximize a well-defined objective in a simple and reasonable setting. Perhaps even more puzzling is that a solution to the example will exist if S_0 is sufficiently small. In other words, there is no optimal way to allocate the endowment S_0 if it is sufficiently large. Since Yaari's (1964) work, a number of follow-up and refinement studies have appeared in both the economics and mathematics literatures, e.g., Aumann and Perles (1965), Kumar (1969), Abrham (1970), Artstein (1974, 1980), and Ioffe (2006). These studies offer a variety of sufficient conditions to guarantee the existence of a solution to the variational problem.

While previous studies have focused on the special case where $j(t) = 0$, $m(t) = 0$, and $f(t) = 1$, the existence problem for the general model (1)–(5) has not yet been studied in the literature. The objective of this paper is to develop sufficient conditions for the existence of a solution to the general model (1)–(5) under the assumption that $\alpha(1) = 0$. As will be explained later, $\alpha(1) = 0$ is a special but not arduous restriction. When the condition holds, a precise existence result can be obtained. The solution to (1)–(5), if it exists, possesses a distinctive feature which can be utilized to generate a simple sufficient condition that guarantees the existence of a solution to the optimal allocation problem. When applied to the special case where $j(t) = 0$, $m(t) = 0$, and $f(t) = 1$, the sufficient condition is substantially simpler than the existing ones in the literature. The existence results reveal why (1)–(5) may not have a solution and whether the existence problem depends on the presence of $j(t)$, $m(t)$, and $f(t)$. In addition, the analysis provides a complete solution to the puzzle raised by Yaari's (1964) counter-intuitive example. While there are many general existence theorems for optimal control problems in the literature (e.g., Cesari, 1983), the relatively simple and explicit structure of this class of optimal allocation problems commensurately deserves a simple and direct existence result.

The plan of the paper is as follows. Section 2 presents the assumptions and three economic examples for (1)–(5). Section 3 investigates the existence problem and provides a series of analytical examples with a closed-form solution to illustrate the usefulness of the existence theorems. Section 4 discusses the role of several major assumptions in the existence results and explores the consequences if the assumptions are relaxed. Section 5 concludes the paper.

2. Assumptions and examples

The following is a list of assumptions on the functions in (1)–(5):

A1. $\alpha(t)$, $m(t)$, $j(t)$, and $f(t)$ are continuously differentiable, $m(t) \geqslant 0$, $j(t) \geqslant 0$, $f(t) > 0$, $\alpha(t) > 0$ for $t \in (0, 1)$, $\alpha'(t) \leqslant 0$, $\alpha(0) = 1$, and $\alpha(1) = 0$.
A2. $g(c)$ is twice continuously differentiable, $g'(c) > 0$, and $g''(c) < 0$.
A3. $S(t)$ is piecewise continuously differentiable.

Let $(c^*(t), S^*(t))$ denote the optimal solution to (1)–(5). From (4) and (5),

$$S^*(t) = e^{\int_0^t j(x)\,dx}\left\{S_0 + \int_0^t e^{-\int_0^z j(x)\,dx}[m(z) - f(z)c^*(z)]\,dz\right\}, \quad t \in [0,1]. \tag{6}$$

The following lemma follows readily from the setup of the model.

Lemma 1. *If the solution $(c^*(t), S^*(t))$ exists, it must be unique and*

$$S^*(1) = 0. \tag{7}$$

Proof. The solution, if it exists, must be unique because (1) is strictly concave in c, the set of feasible $c(\cdot)$ is convex, and (4) is linear in $c(t)$ and $S(t)$. To prove (7), suppose the contrary that $S^*(1) > 0$. By the left-continuity of $S^*(t)$ at $t = 1$, there exist $\sigma_1 > 0$ and $\sigma_2 > 0$ such that $S^*(t) > S^*(1) - \sigma_1 > 0$ for $t \in (1 - \sigma_2, 1)$. Consider the consumption path $c^{**}(t) = c^*(t)$ for $t \in$