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# Small noise methods for risk-sensitive/robust economies

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ABSTRACT

We provide small noise expansions for the value function and decision rule for the recursive risk-sensitive preferences specified by Hansen and Sargent (1995), Hansen et al. (1999), and Tallarini (2000). We use the expansions (1) to provide a fast method for approximating solutions of dynamic stochastic problems and (2) to quantify the effects on decisions of uncertainty and concerns about robustness to misspecification.


Small noise expansions justify standard measures of risk aversion and precaution in models of decision-making under uncertainty (see Pratt, 1964). In a static or a two-period decision problem, risk aversion measures are computed as amounts that compensate a consumer for accepting risk. These measures become arbitrarily accurate when the variance of the risk approaches zero. Similarly, measures of precautionary saving are derived by approximating the response of savings to increased risk, where again arbitrary accuracy is achieved when the variance becomes sufficiently small. One aim of the present paper is to compute such measures for a class of discounted, infinite-horizon control problems. Another aim is to seek measures of aversion to bearing model uncertainty.

We study small noise expansions for discrete-time infinite horizon control problems with risk sensitivity or equivalently with a concern about robustness to model misspecification. We follow Epstein and Zin (1989) and model the preferences of the decision-maker recursively.[1] Hansen and Sargent (1995) showed that for linear–quadratic, Gaussian control problems, a recursive formulation of risk sensitivity preserves the tractability of risk-sensitive control theory and for infinite-horizon control problems delivers decision rules with time-invariant risk adjustments. The solution of the risk-sensitive control problem of Hansen and Sargent (1995) is identical to the solution of a particular type of robust control problem. That equivalence allows us to interpret our expansions as providing a way to quantify the effects of concerns about robustness to model misspecification.

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[1] In the control theory literature, infinite-horizon, risk-sensitive problems have been studied mostly in models without discounting (e.g. see Whittle, 1990; Glover and Doyle, 1988; Runolfsson, 1994).

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Our small noise expansions are closely related to expansions in the control-theory literature [e.g. see Fleming and Souganidis, 1986; James, 1992; James et al., 1994; Fleming and Yang, 1994, and especially Campi and James, 1996] but differ from expansions that are typically used in economics. Judd and Guu (1993, 1997) and Judd (1996, 1998) explore small noise expansions in the shock standard deviation in conjunction with expansions in the state variables about a deterministic steady state. Because we expand with respect to the shock variance only, we can approximate around a deterministic trajectory. Some recent work in economics has contributed expansions in other parameters. For example, Kogan and Uppal (2003) consider an expansion in a risk aversion preference parameter. As we will see, our first-order expansion can be re-interpreted as a joint expansion in the shock variance and a parameter governing either risk sensitivity or robustness.

Several recent papers have considered alternative perturbation methods for risk-sensitive (robust) problems. Chen and Zadrozny (2003) obtain a fourth-order expansion for problems in which preferences are quadratic. Following Kogan and Uppal (2003), Trojani and Vanini (2002) obtain an expansion in an analog of a risk-aversion parameter. Hansen et al. (2007, 2008) also extend Kogan and Uppal (2003) by expanding in the reciprocal of the intertemporal elasticity of substitution.

We focus on using small noise expansions for two different purposes:

1. To provide a fast method for approximating solutions of dynamic stochastic problems.
2. To quantify the effects on decision of uncertainty and concerns about robustness to misspecification on optimal decisions.

In the stochastic growth model, we show that a small noise expansion around a deterministic trajectory (which we refer to as the path method) typically provides a significantly more accurate solution than a small noise expansion around a steady state (which we refer to as the steady state method). We also provide examples of how small noise expansions can be used to measure the long-run effects of robustness on capital accumulation, consumption, and asset returns.

The remainder of this paper is organized as follows. In Section 1 we describe the risk-sensitive control problem that is the focus of this paper. In Section 2 we give a heuristic derivation of the Campi–James small noise value function expansion appropriately modified for the infinite-horizon, recursive formulation of risk-sensitive preferences of Hansen and Sargent (1995). In Section 3, we provide a first-order small noise approximation for the decision rule. In Section 4, we describe an algorithm for computing expansions and unconditional expectations. In Section 5, we describe expansions for asset prices. In Section 6, we provide two simple examples where the expansion can be computed analytically. In Section 7, we discuss the relationship between the path expansions described in this paper with the typical steady state expansion used in economics and in Section 8 compare their accuracy for a stochastic growth model. In Section 9 we investigate the robustness interpretation of risk-sensitive preferences. In Section 10 we study the implications of second-order expansions in a stochastic growth model. In Section 11 we extend the expansions to the Kreps–Porteus CES specification. Our conclusions are in Section 12.

## 1. A risk-sensitive control problem

A decision-maker has an infinite horizon and an increasing sequence of sigma algebras (information sets) $\{\mathcal{F}_t : t=0,1,\ldots\}$. Let $i_t$ denote a time $t$ control vector such as investment; $i_t$ must be adapted to the conditioning information set $\mathcal{F}_t$. There is a sequence of i.i.d. shocks $\{w_t : t=0,1,\ldots\}$ in which $w_t$ is normally distributed with mean zero and covariance $I$. We take this process to be adapted to $\{\mathcal{F}_t\}$, with $w_{t+1}$ independent of $\mathcal{F}_t$. Let the state vector be denoted $x_t$ where the process $\{x_t : t=0,1,\ldots\}$ is also adapted to $\{\mathcal{F}_t\}$. The state vector evolves as

$$x_{t+1} = A(x_t,i_t) + \sqrt{\epsilon}\Lambda(x_t)w_{t+1} \tag{1}$$

with the initial value $x_0$ known ($x_0$ is in $\mathcal{F}_0$). The parameter $\epsilon$ enables "small noise" expansions of the value function and the decision rule.

As in Hansen and Sargent (1995), Hansen et al. (1999), Tallarini (2000), and Anderson et al. (2003), we model preferences of the decision-maker using the recursion:

$$U_t = u(x_t,i_t) + \mathcal{R}_t(\beta U_{t+1})$$

where

$$\mathcal{R}_t(\beta U_{t+1}) \equiv -\frac{1}{\sigma}\log E[\exp(-\sigma\beta U_{t+1})|\mathcal{F}_t]$$

The function $\mathcal{R}_t$ makes an additional risk adjustment to the continuation value function and is the vehicle by which we introduce risk sensitivity. As emphasized by Hansen and Sargent (1995), the log−exp specification of the recursion provides a bridge between risk-sensitive control theory and a more general recursive utility specification of Epstein and Zin (1989). The degree of risk sensitivity is quantified by $\sigma$. When $\sigma=0$, we have the usual Von Neumann–Morgenstern form of state additivity: $\mathcal{R}_t(\beta U_{t+1}) \equiv \beta E(U_{t+1}|\mathcal{F}_t)$. Values of $\sigma$ greater than zero increase aversion to risk *vis a vis* the Von Neumann–Morgenstern specification.