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Static and dynamic analysis of beam assemblies using a differential system on an oriented graph

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ABSTRACT

Many systems in engineering, theoretical physics and other domains of natural sciences can be investigated using a linear mathematical model having the character of a differential system defined within a given network. This network may consist of one-dimensional elements characterised by local coordinate systems. These elements (recti- or curvilinear) are interconnected at nodes, through which energy, mass and stiffness properties of the elements are transmitted as a function of time. The system as a whole is generally determined by some boundary conditions or assumed to be interconnected with other subsystems. Elements of the system are considered to have continuously distributed parameters (mass, stiffness, conductivity, etc.). External energy may be supplied through boundary conditions or by excitation of elements at nodes. The problem of the system's response, or a relevant eigenvalue problem, can be understood as a problem of a differential system on an oriented graph. This graph is a corresponding geometrical representation of the system investigated, where elements of the graph represent individual beams of the system. Therefore the physical part of the problem is fully included in the original differential system, but without any indication of its domain shape. As illustrations of this theoretical study, the conventional Slope Deflection Method (SDM), developed in the past for statics and later for dynamics of continuous frames are outlined in this paper, along with some illustrations from other branches. It should be noted that the character of the resulting algorithm is similar to the FEM if special macro-elements provided by direct solution of the relevant differential system are used. High numerical stability of the approach used here is a significant strength in comparison with other procedures. This follows from the principal attributes of the proposed method. Easy implementation of the theory into existing software packages is possible.

1. Introduction

Various approaches are applicable to analyse the response, modal characteristics and other attributes of structures in civil, mechanical or other branches of engineering. Structures that can be analysed using a linear mathematical model based on a system of one-dimensional rectilinear or curvilinear elements are considered here. The system generates a 2D or 3D framework with continuously distributed parameters (mass, stiffness, conductivity, etc.). The elements are interconnected at nodes, making a kinematically determinate system. Some nodes may be fixed by imposed external factors that represent the boundary conditions. Determining a system response or modal characteristics can be considered as a problem of a differential system on an oriented graph, which is an adequate geometrical representation of the system to be investigated. It should be noted that the coefficients of such a differential system are functions of the local length coordinate, which is generally piecewise continuous.

This idea applies to any physical problem which can be described by a relevant linear differential system in terms of length coordinates and time, and is therefore applicable in areas such as heat propagation, flow in a power-piping system, electromagnetic fields in a conducting network, various optical problems, chemical reactions in particular piping systems, and relevant biological, social, logistical and other processes. The theoretical basis of this approach dates back to the 1700s and work of Leonhard Euler as reported in [1]. It gained popularity in early sixtieth, when abstract formulation of the general framework geometry using an oriented graph was outlined due to needs of development of the first computer systems for linear static analysis of frameworks in civil engineering [2]. Strengths of this approach were obvious eliminating the need of individual treatment of each structure. The linear graph theory was recognised to be highly relevant in development of

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multibody dynamics, preparing a basis for computer systems intended for the formal analysis of complicated dynamic systems in mechanical engineering and robotics [3,4]. The graph theory in this area attracted later mathematicians involved in the topology, where abstract theoretical background has been developed. Many results contributed in early period and obtained by mathematicians have been later employed to formulate many special engineering tasks [5] and the graph theory still keeps its popularity today, see, e.g. [6,7].

All these formulations, however, suffer from one major deficiency: they are tightly linked to the physical nature of the problem. Hence the need to involve physical properties in all further derivation aimed at formulation of the conditions which must be fulfilled in nodes of the graph. In such a case unpleasant problems of ambiguity can occur especially when the differential operator is complicated or interaction of several physical fields are considered. On the other hand, it is obvious that discussion concerning a physical character of the problem should finish at the moment when the differential system is formulated without specification of a particular physical system. Therefore for instance in mechanics of frameworks no equations of equilibrium of forces and moments in nodes should be formulated or energy balance in nodes when thermodynamic problem is investigated.

This drawback, however, can be eliminated on a purely mathematical basis. The basic idea follows from an examination of a symmetric character of the relevant differential operator. The condition of the existence of the unique solution is symmetry of the differential operator. Formal verification of this property via repeated integration per partes leads to conditions that all emerging out-of-integral terms should vanish, see Section 2. These conditions provide all necessary equations following (i) from continuity of main quantities consisting of the response lower derivatives $(0,\ldots,n-1$, where $2n$ is order of the operator) and their combinations and (ii) from equilibrium of natural quantities containing response derivatives of higher order $(n,\ldots,2n-1)$. A subsequent analysis of the particular form of such conditions shows that they represent respective physical conditions of the underlying model, e.g. equilibrium of forces and moments acting in a node in mechanics. Note that the above operation does not lead to weak solution as it is observed in the FEM analysis. Basic principles of the procedure presented here are fully different and produce the classical strong solution.

The presented procedure provides an abstract mathematical basis covering a wide range of particular models. The physical principles are applied only for assembly of the differential system. Subsequent solution is performed by mathematical operations dealing with the differential system on a special domain – an oriented graph. Physical conditions joining individual parts of the model result from the conditions for the differential operator to keep symmetry. According to the best knowledge of the authors, such a procedure has not been used before. Its indisputable advantage is that it allows immediate applications in areas beyond the mechanics or even beyond physics. Relevant examples are given below.

The first attempt at general formulation of this type was outlined to some extent in [8], and later in [9]. With regard to problems of framework dynamics, a number of applications of this general approach were discussed in [10,11].

In civil engineering, similar results can be obtained by using the classical Slope Deflection Method (SDM) which was originally developed for static framework problems based on unit deflections method [12], and later for steady state dynamic problems [13]. These very widely used conventional methods in practical engineering which are degenerate and special cases of the theory presented in this paper. It should be recognised that various aspects of the classical methods are still under development, see an historical overview [14], and a detailed comparison with the Force Method [15].

A system consisting of beams can be analysed using FEM and there are software packages available for this type of analysis, see e.g. [16,17]. Every beam is replaced by the appropriate number of finite elements, and finally the necessary information about the response at the nodes is obtained. Nevertheless, to be able to offer an alternative to FEM using the approach outlined here is useful and practical for many reasons because the solution obtained is very much like an analytical solution, and consequently it provides a more accurate answer and better insights. This also enables a deeper physical interpretation of results, including a more reliable assessment of the sensitivity of the system to small differences in the parameters.

Note that the calculation of a particular case using the proposed approach is usually significantly faster than the FEM because the number of nodes needed to achieve a comparable accuracy is dramatically lower. Furthermore, an appropriate procedure can be easily implemented into a computer program. It is worth noting that a solution worked out using the algorithm outlined here could be used as an indicative guidance when a specialised finite element is being developed or when a global algorithm is tested. Of particular significance is the fact that both managing the theoretical aspect of the problem and its software implementation are straightforward.

Another verification of the generality of framework response solutions using differential systems of motion are the published works that have been appearing for several decades dealing with beams with supplementary masses and isolated cracks. The published procedures are usually quite complex and convoluted and often susceptible to loss of numerical stability; see e.g. [18]. These difficulties mostly disappear using the procedures outlined below. The reason lies in the implementation strategy of continuity constraints at nodes that follows from the rigorous theoretical steps.

This paper is basically based upon the earlier work of Náprstek and Fischer [9], but with significant enhancement of the idea through formal and informal discussions. This is true for all sections. The theoretical background is based on that of [9], but the mathematical formulation has been specified and supplemented. A deeper relationship with the rigorous theory of graphs has been amended in several places. A number of references, figures and examples, together with their detailed mathematical formulation, have been added.

2. General considerations

Although mechanics and other related disciplines in physics have not paid particular attention to the employment of the theory of graphs for investigating relevant problems, many special studies have appeared in the mathematics literature in the form of papers and monographs. This applies also to systems of ordinary differential equations, where the special properties of graphs can help in the understanding of many unconventional problems; see for instance papers [19–21] and monograph [22], where many problems of differential systems on oriented graphs of various types have been discussed. Special aspects regarding differential systems on graphs from a mathematical standpoint have been investigated in many papers, in particular boundary and eigenvalue problems using networks; see for instance [23–25]. The general background of graph theory can be found in a number of monographs, see for instance [26,27].

Throughout this paper it is assumed that a graph representing a given problem is a finite, simple, connected and oriented metric graph. Graph $G(N,E)$ is finite if its set of nodes N and edges E are finite. Edges and nodes of a graph are related through the incidence