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Numerical evaluation of nearly hyper-singular integrals in the boundary element analysis

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ABSTRACT

Accurate evaluation of nearly singular integrals plays an important role in the overall accuracy of the boundary element analysis. The paper presented here reviews some numerical techniques used currently to calculate such integrals in the boundary element method (BEM). Some incorrect algorithms published before are discussed and a new numerical technique to calculate nearly singular integrals with hyper-singularities is developed. The accuracy and efficiency of the method are demonstrated through several examples that are commonly encountered in the applications of the BEM. Comparison of this method with some of the existing methods is also presented. It is shown that, for the evaluation of nearly hyper-singular integrals, several orders of magnitude improvement in relative error can be obtained using the current method compared to most of existing methods.

1. Introduction

Accurate evaluation of nearly singular integrals [1–4] is an important consideration in an implementation of the boundary element method (BEM). These integrals are *nearly singular* in the sense that the calculation point is approaching towards, but not on, the boundary element. Theoretically, these integrals are regular since the values of the integrands are always finite. However, instead of remaining smooth, the integrand develops a sharp peak as the calculation point moves closer to the boundary. Accurate evaluation of such integrals faces considerable difficulties because neither the traditional Gaussian quadrature nor the methods designed for singular integrals can be employed [5,6].

Tremendous effort has been devoted to deriving convenient integral forms or sophisticated computational techniques for the accurate evaluation of such integrals. The methods developed so far include, but are not limited to, the element subdivision method [7–9], analytical or semi-analytical methods [10–12], and various coordinate transformations [13–20]. Impressive results obtained from these techniques have been demonstrated on various examples. However, a number of drawbacks still remain and mainly include the fact that some techniques are restricted to nearly singular integrals defined on linear geometry elements while other are tailored to some certain kind of integrals.

In this paper, a new approach based on coordinate transformations for evaluating nearly singular integrals is introduced. The aim of the present paper is at developing a general method that is suitable for a wide range of nearly singular integrals. Particular focus is on integrals with near hyper-singularities and beyond. The outline of the rest of the paper is as follows. A brief summary of the current methods for the calculation of nearly singular integrals is given in Section 2. The nearly singular integrals arising in two-dimensional (2D) boundary element analysis are described in Section 3. The merits and drawbacks of several coordinate transformations, namely the sinh transformation [15], the exponential transformation [21] and rational transformation [14], are discussed in Sections 4 and 5, respectively. The new method and its numerical implementation are introduced in Section 6. Results on several benchmark examples obtained from various methods are presented and compared next. Finally, the conclusions and remarks are provided in Section 8.

2. Brief summary of nearly singular integration techniques

An effective technique to handle nearly singular integrals is critical to achieve efficient, accurate boundary element analysis. Many papers have been published and several efficient techniques have been suggested and used with varying degree of success. The

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numerical techniques used currently to calculate such integrals can roughly be summarized into the following categories.

1. *Element subdivision methods*: Element subdivision method was first proposed by Lachat and Watson [7] and was used for three-dimensional (3D) elasticity problems. The method is appealing, stable, and accurate but is costly because the number of sub-elements and their sizes are strongly dependent on the order of the near singularity, the ratio of the shortest distance from the singular point to the element, and the dimension of the element [8,22].
2. *Analytical and semi-analytical methods*: By adopting local polar co-ordinates, Cruse and Aithal [23] provided a highly accurate algorithm for both singular and nearly singular integrals. By constructing a tangent plane at the singular point or at the closest node on the element where the integral is taken, the singular and nearly singular integrals can be converted into the sum of a regular integral which can be calculated analytically. The method is accurate but it is not available for a general distributed function and cured boundary surface because closed-form integrations of the kernel function and boundary function are not possible. In a series of papers published by Niu and Zhou [11,12], asymptotic expansion of the kernel function with respect to the local co-ordinates is suggested such that the divergent part of the singular integral can be separated from the remaining regular integrals. The separated singular terms are then calculated analytically.
3. *Simple/Particular solution methods*: The idea of the method benefits from the regularization ideas (rigid-body movement methods) for evaluating singular integrals [24], which use some simple solution of the original partial differential equation to compute the nearly singular integrals indirectly [25]. This numerical technique is simple and can be used for curved boundary surface, but the accuracy of the method is not satisfactory when the calculation point is very close to the boundary element.
4. *Adaptive Gaussian quadrature method*: A special Gaussian type numerical integral was proposed by Pina et al. [26] and Lutz [27] to calculate the singular and nearly singular integrals. A set of special weights and Gaussian points were derived based on the form of the integrand and the distance between the calculation point to the boundary element.
5. *Modified boundary integral equation*: This kind of method avoids calculating the nearly singular integrals by establishing new regularized boundary integral equations, such as the virtual boundary element method [28,29] and the method of fundamental solutions [30]. This is done to place a fictitious boundary slightly outside the problem domain and to avoid the singularity of the fundamental solutions. The determination of the distance between the real boundary and the fictitious boundary is based on experience and therefore troublesome.
6. *Coordinate transformation methods*: The most popular techniques used currently to calculate the nearly singular integrals are various nonlinear transformations. The key idea of the methods is to remove or smooth out the near singularities of the integrand using a coordinate transformation before the conventional Gaussian quadrature is applied. The methods developed so far include, but are not limited to, the cubic polynomial transformation [13], bi-cubic transformation [31], rational transformation [14], optimal transformation [17], distance transformation [2], sigmoidal transformation [32], sinh transformation [15,33–35] and exponential transformation [21,36]. Impressive results obtained from these transformations have been demonstrated on various examples. It is, however, difficult to find one particular transformation that is effective for a wide range of nearly singular integrals.

In a recent study conducted by Johnston et al. [33,37], a number of transformation techniques have been compared. It was concluded that the sinh transformation is very successful and is either superior or equivalent to most of existing methods in terms of overall accuracy and stability. However, as will be pointed out in the following, the method is accurate for nearly weakly/strongly singular integrals, but is not able to remove the singularities of nearly hyper-singular integrals.

3. The nearly singular integrals

The nearly singular integrals arising in 2D boundary element method can be expressed as the following generalized forms:

$$I_1 = \int_{-1}^{1} f(\xi)\log r(\xi) d\xi, \tag{1}$$

$$I_2 = \int_{-1}^{1} f(\xi)\frac{1}{r^{2\alpha}(\xi)} d\xi, \quad \alpha > 0, \tag{2}$$

where $\xi \in [-1,1]$ is the local intrinsic coordinate, the function $f(\xi)$ denotes a low-order polynomial which may consist of the Jacobian of the transformation from some arbitrarily curved element Γ to line interval $[-1,1]$, shape functions used to interpolate the physical solution and/or the term which arises from taking the derivative of the boundary element kernel. Here, r is the distance from the calculation point to the integral element Γ.

An implementation of the BEM requires the accurate evaluation of the above-mentioned integrals. When the calculation point is far from the boundary element under consideration, a straightforward application of Gaussian quadrature suffices to evaluate such integrals. When the calculation point is on the integral element, the integrand becomes singular and many direct and indirect algorithms have been developed and used successfully [38,39].

A class of integrals which lies between these two extremes is that of the *nearly singular integrals*. Here, the calculation point is close to, but not on, the element and the integrals, theoretically, are regular since the values of their integrands remain finite at all points. However, instead of remaining flat, the magnitude of the integrand may be quite large as the calculation point approaching towards the integral element. The evaluation of such integrals faces considerable difficulties because neither the conventional Gaussian integration nor the methods designed for singular integrals are applicable here. Similar to singular integrals, when the calculation point moves closer to the boundary, the integral (1) has a nearly weak singularity of order $(\ln r)$ in 2D problems, the integral (2) has a nearly strong singularity of order $\left(\frac{1}{r}\right)$ with $\alpha = \frac{1}{2}$, while the integral (2) present a nearly hyper-singularity of order $\left(\frac{1}{r^{\alpha}}\right)$ with $\alpha > \frac{1}{2}$.

When the geometry is approximated using linear element, the distance function $r^2(\xi)$ in Eqs. (1) and (2) can be expressed as $r^2(\xi) = (\xi - a)^2 + b^2$, where the parameters $a(a \in [-1,1])$ and $b(b > 0)$ represent the position of the nearly singular point and

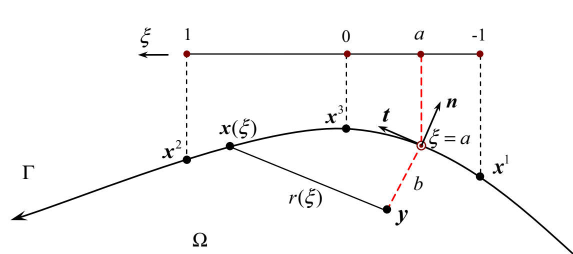

Fig. 1. Geometry of a parabolic boundary element.