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Quasi-linear integrability ☆

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Abstract

Applied researchers often work with demand systems that do not depend on income, with the implicit assumption that preferences are quasi-linear and income sufficiently large. The classic approach to the integrability of demand does not readily apply in this case. Adopting a much simpler approach that is based on integrating the vector field defined by the demand system and on duality, we provide necessary and sufficient conditions for the quasi-linear integrability of such (continuous) demand systems. We also derive results on the associated utility function and its domain, and provide an application to the analysis of demand systems in the presence of measurement errors.

JEL classification: D01; D11

Keywords: Integrability; Quasi-linear preferences; Duality

1. Introduction

In this paper, we provide necessary and sufficient conditions for the quasi-linear integrability of a demand system. That is, we analyze the classic *integrability problem* for the case when observed demand does not vary with income and consumers have quasi-linear preferences.

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In industrial organization and other applied fields of microeconomics, researchers typically focus on partial equilibrium settings and therefore often work with demand systems that depend on prices but not on income. The implicit assumption they make is that preferences are quasi-linear and consumer income sufficiently high, so that not all income is spent on the goods offered in the market under consideration. It remains unspecified what demand would look like if income were so low that no outside good would be consumed.

The classic treatment of the integrability problem is due to Hurwicz and Uzawa (1971). Their approach consists of two key steps. First, they integrate the demand system to obtain an income compensation function, using tools from the literature on differential forms. Second, inverting the demand system and using the income compensation function, they construct a utility function, defined over the range of the demand system. The second step has been subsequently improved by Jackson (1986), who uses duality theory to obtain an upper semi-continuous utility function defined over the entire non-negative orthant. However, this approach does not apply (and does not extend easily) to the "quasi-linear case" in which demand is not a function of income. In particular, the existing approach relies on (i) the differentiability of demand at all prices and income levels, (ii) the boundary condition that, for any price vector, demand is zero whenever income is zero, and (iii) the non-negativity of demand.

How can the existing approach and results be extended to the quasi-linear case in which the specified demand is not a function of income?[1] One possible solution would involve allowing for negative consumption of the outside good whenever income is sufficiently low. However, this would obviously violate conditions (ii) and (iii). An alternative solution would involve extending the demand system in some way to account for cases in which income is sufficiently low. However, this would necessarily introduce a non-differentiability, violating condition (i).[2]

In this paper, we develop a much simpler approach for continuous demand systems that do not depend on income. First, we show the existence of a (candidate) indirect subutility function by integrating the vector field defined by the demand system. Second, we construct a direct quasi-linear utility function, using duality theory. We show that quasi-linear integrability amounts to the demand system being a conservative vector field that satisfies the law of demand. If demand is continuously differentiable, then this condition is equivalent to the symmetry and negative semi-definiteness of the substitution matrix of the demand system.[3] Moreover, under these conditions, the associated indirect utility function is unique up to an additive constant, and continuously differentiable, implying that any induced change in money-metric consumer welfare is uniquely pinned down. Under the same conditions, the demand system can be derived from a monotone, concave and upper semi-continuous subutility function defined over some set X that contains the comprehensive convex hull of the range of the demand system. Furthermore, the resulting subutility function is continuous in the interior of X.

We obtain additional results in Section 4. There, we provide results on the shape of X, on the maximality of X, derive conditions under which the subutility function is continuously differen-

[1] See Amir et al. (2017) for a treatment of quasi-linear integrability when demand is linear.

[2] Hosoya (2016, Section 3) proves an integrability theorem for demand systems that are defined over an open cone of the positive orthant. While it is possible to apply his results to study quasi-linear integrability, his construction relies on the demand system being continuously differentiable and surjective, an assumption which we do not need to make.

[3] The fact that the maximization of a quasi-linear utility function with a negative definite Hessian matrix delivers a demand system with a symmetric and negative semi-definite substitution matrix is well-known (see, e.g., Vives, 2000, Section 3.1). In this paper, we fully characterize the set of continuous demand systems that are derivable from quasi-linear utility maximization.