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On the similarity of meshless discretizations of Peridynamics and Smooth-Particle Hydrodynamics

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ABSTRACT

This paper discusses the similarity of meshless discretizations of Peridynamics and Smooth-Particle-Hydrodynamics (SPH), if Peridynamics is applied to classical material models based on the deformation gradient. We show that the discretized equations of both methods coincide if nodal integration is used. This equivalence implies that Peridynamics reduces to an old meshless method and all instability problems of collocation-type particle methods apply. These instabilities arise as a consequence of the nodal integration scheme, which causes rank-deficiency and leads to spurious zero-energy modes. As a result of the demonstrated equivalence to SPH, enhanced implementations of Peridynamics should employ more accurate integration schemes.

1. Introduction

The Peridynamic theory, originally devised by Silling [1], is a nonlocal extension of classical continuum mechanics, which is based on partial differential equations. Since partial derivatives do not exist on crack surfaces and other singularities, the classical equations of continuum mechanics cannot be applied directly when such features are present. In contrast, the Peridynamic balance of linear momentum is formulated as an integral equation, which remains valid in the presence of material discontinuities. Therefore, the Peridynamic theory can be applied directly to modeling both bulk *and* interface properties, using the same mathematical model. Additionally, Peridynamics is readily implemented in a meshless formulation, which facilitates the simulation of large deformations when compared to the traditional mesh-based Finite-Element method used for simulating solid mechanics in the classical continuum theory [2].

With these desirable features, Peridynamics has received considerable attention by researchers interested in numerically describing fundamental crack growth and failure effects in brittle materials [3–6]. However, the scope of the original Peridynamic formulation included only so-called *bond-based* models [7] which were limited to a fixed Poisson ratio for linear isotropic materials and could not describe true plastic yielding. Further development of the theory led to *state-based* models [8], which can, in principle, describe all classic material behavior. In particular, the state-based theory offers a route to approximating the classical deformation gradient, which can be used to obtain a classical stress tensor. The stress tensor can then be converted into nodal forces using a Peridynamic integral equation. This promising situation, i.e., the ability to use classical material models with a method that remains valid at material discontinuities, has prompted a number of studies where plastic yielding, damage, and failure was simulated using this new meshless method [9,10].

However, there exists a demand for studies which compare Peridynamics to other meshless methods. While the theoretical correspondence of Peridynamics with classical elasticity theory has been established [11], no information is available on the accuracy of the discretized Peridynamic expression suitable for computer implementation. Relevant questions that need to be addressed include, but are not limit to, whether linear fields can be exactly reproduced by the discretized theory, and what the order of convergence is when Peridynamic solutions are compared against exact results. Little is known about how common problems encountered with other meshless methods, e.g., the tensile instability [12] or the rank deficiency problem [13] affect Peridynamics. As an exception to this general observation, Bessa et al. [14] have published a study which demonstrates the equivalence of state-based Peridynamics and the reproducing Kernel Particle Method (RKPM), if nodal integration is used. However, no published studies are available to the best of this author's knowledge, which compare Peridynamics to other meshless methods.

The goal of our paper is to elucidate on some properties of Peridynamics with respect to other meshless methods. In particular, we establish that the discrete equations of the Peridynamic formulation using classical material models is identical to a very

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well-known meshless technique: Smooth Particle Hydrodynamics (SPH) in the Total-Lagrangian formulation. This equivalence facilitates understanding of Peridynamics using the large body of literature already published for other meshless methods, see e.g. [15,16]. The key observations of our work address two issues: (i) Discretizations of Peridynamics directly arrive at correct equations which conserve linear and angular momentum. These features can only be obtained in SPH by assuming *ad hoc* corrections such as explicit symmetrization. (ii) All of the problems that apply to collocation-type particle methods also apply to Peridynamics if this theory is discretized using nodal integration.

The remainder of this paper is organized as follows: we begin by deriving the fundamental expressions of the SPH approximation including the most important corrections for this method, which allow it to be used with the minimal level of accuracy required for solid mechanics simulations. Then, the essential Peridynamic expressions required for simulating classical material models are derived. Building on this foundation, the equivalence of SPH and this particular variant of Peridynamics is shown. Finally, the implications of this observation are discussed and Peridynamics (with classical material models) is characterized using the established terminology encountered in the SPH literature.

2. Total Lagrangian SPH

Smooth Particle Hydrodynamics [17] was originally devised as a Lagrangian particle method with the smoothing kernel moving with the particle, thus redefining the interaction neighborhood for every new position the particle attains. In this sense, the kernel of the original SPH formulation has Eulerian character, as other particles move through the interaction neighborhood. The tensile instability [12] encountered in SPH, where particles clump together under negative pressure conditions, has been found to be caused by the Eulerian kernel functions [13]. Consequently, Total Lagrangian formulations were developed [18–20], which use a constant reference configuration for defining the interaction neighborhood of the particles. see Typically, the initial, undeformed configuration is taken for this purpose. In the following, this concept and the associated nomenclature is briefly explained with the limited scope of obtaining SPH expressions that are to be compared with the Peridynamic expressions. For a more detailed derivation, the reader is referred to the works cited above.

2.1. Total Lagrangian formulation

In the total Lagrangian formulation, conservation and constitutive equations are expressed in terms of the material coordinates $\boldsymbol{X}$, which are taken to be the coordinates of the initial, undeformed reference configuration. A mapping between the current coordinates, and the reference coordinates describes the body motion at time t:

$$\boldsymbol{x} = \phi(\boldsymbol{X}, t), \tag{1}$$

Here, $\boldsymbol{x}$ are the current, deformed coordinates and $\boldsymbol{X}$ the reference (Lagrangian) coordinates. The displacement $\boldsymbol{u}$ is given by

$$\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{X}, \tag{2}$$

The conservation equations for mass, impulse, and energy in the total Lagrangian formulation are given by

$$\rho J = \rho_0 J_0, \tag{3}$$

$$\ddot{u} = \frac{1}{\rho_0}\nabla_0 \cdot \boldsymbol{P}, \tag{4}$$

$$\dot{e} = \frac{1}{\rho_0}\dot{\boldsymbol{F}} : \boldsymbol{P}^T, \tag{5}$$

where J and J_0 are the current and initial Jacobian determinants. ρ is the current mass density and ρ_0 is the initial mass density, $\boldsymbol{P}$ is the nominal stress tensor (the transpose of the first Piola–Kirchhoff stress tensor), e is the internal energy, ∇_0 is the gradient or divergence operator expressed in the material coordinates, and $\boldsymbol{F}$ denotes the deformation gradient,

$$\boldsymbol{F} = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\boldsymbol{X}} = \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}\boldsymbol{X}} + \boldsymbol{I}, \tag{6}$$

2.2. The SPH approximation

The SPH approximation for a scalar function f in terms of the Lagrangian coordinates can be written as

$$f(\boldsymbol{X_i}) = \sum_{j\in\mathcal{S}} V_j^0 f(\boldsymbol{X_j}) W_i(X_{ij}). \tag{7}$$

The sum extends over all particles within the range of a scalar weight function W_i, which is centered at position $\boldsymbol{X_i}$ and depends on the distance between coordinates $\boldsymbol{X}_i$ and $\boldsymbol{X}_j, X_{ij} = \|\boldsymbol{X_j} - \boldsymbol{X_i}\|$. V^0 is the volume associated with a particle in the reference configuration. The weight function is typically chosen to be radially symmetric and have compact support, i.e., it includes only neighbors within a certain radial distance. This domain of influence is denoted S.

The SPH approximation of a derivative of f is obtained by operating directly with the gradient operator on the kernel functions,

$$\nabla f(\boldsymbol{X_i}) = \sum_{j\in\mathcal{S}} V_j^0 f(\boldsymbol{X_j}) \nabla W_i(X_{ij}), \tag{8}$$

where the gradient of the kernel function is defined as follows:

$$\nabla W_i(X_{ij}) = \left(\frac{\mathrm{d}W(X_{ij})}{\mathrm{d}X_{ij}}\right)\frac{\boldsymbol{X}_j - \boldsymbol{X}_i}{X_{ij}}. \tag{9}$$

The conditions for the zeroth- and first-order completeness of the SPH approximation are stated as follows:

$$\sum_{j\in\mathcal{S}} V_j^0 W_i(X_{ij}) = 1 \tag{10}$$

$$\sum_{j\in\mathcal{S}} V_j^0 \nabla W_i(X_{ij}) = 0 \tag{11}$$

In the simple form as stated here, neither of the completeness conditions are fulfilled by the SPH approximation. An *ad hoc* improvement consists in adding Eq. (11) to Eq. (8), such that a *symmetrized* approximation for the derivative of a function is obtained,

$$\nabla f(\boldsymbol{X_i}) = \sum_{j\in\mathcal{S}} V_j^0 \left(f(\boldsymbol{X_j}) - f(\boldsymbol{X_i})\right) \nabla W_i(X_{ij}). \tag{12}$$

The symmetrization does not result in first-order completeness, however, it yields zeroth-order completeness for the derivatives of a function.

2.3. Restoring first-order completeness

In order to fulfill first-order completeness, the SPH approximation has to reproduce the constant gradient of a linear field. A number of correction techniques [21–23] exploit this condition as the basis for correcting the gradient of the SPH weight function,

$$\sum_{j\in\mathcal{S}} V_j^0 (\boldsymbol{X}_j - \boldsymbol{X_i}) \otimes \nabla W_i(X_{ij}) \overset{!}{=} \boldsymbol{I}, \tag{13}$$

where $\boldsymbol{I}$ is the diagonal unit matrix. Based on this expression, a corrected kernel gradient can be defined:

$$\tilde{\nabla} W_i(X_{ij}) = \boldsymbol{L}_i^{-1} \nabla W_i(X_{ij}), \tag{14}$$

which uses the correction matrix $\boldsymbol{L}$, defined as: