Computers and Structures 153 (2015) 112–125

Contents lists available at ScienceDirect

Computers and Structures

journal homepage: www.elsevier.com/locate/compstruc

Dynamic equations for a fully anisotropic piezoelectric rectangular plate

K. Mauritsson [1], P.D. Folkow *

Department of Applied Mechanics, Chalmers University of Technology, SE-412 96 Göteborg, Sweden

ARTICLE INFO

Article history:
Received 18 December 2013
Accepted 18 February 2015
Available online 25 March 2015

Keywords:
Piezoelectric plate
Power series
Asymptotic
Orthotropic
Eigenfrequency
Laminate

ABSTRACT

A hierarchy of dynamic plate equations based on the three dimensional piezoelectric theory is derived for a fully anisotropic piezoelectric rectangular plate. Using power series expansions results in sets of equations that may be truncated to arbitrary order, where each order set is hyperbolic, variationally consistent and asymptotically correct (to all studied orders). Numerical examples for eigenfrequencies and plots on mode shapes, electric potential and stress distributions curves are presented for orthotropic plate structures. The results illustrate that the present approach renders benchmark solutions provided higher order truncations are used, and act as engineering plate equations using low order truncation.

1. Introduction

Piezoelectric materials have been used widely in applications for sensing and actuation purposes in recent years. As piezoelectric sensors and actuators usually are thin in comparison to relevant wavelengths, the analyzes of thin piezoelectric layers, such as beams, plates and shells, have attained considerable research interest. Many references to work on higher-order piezoelectric plate theories prior to 2000 are given in the review article by Wang and Yang [1]. Further references to laminated piezoelectric plates are presented in [2] while [3] presents a classification and comparison among higher-order piezoelectric plate models based on power series expansions. A more recent review article on three dimensional approaches for piezoelectric plates is presented by Wu et al. [4].

Plate theories for various material configurations were developed in the 50's by Mindlin, among which piezoelectric plates were addressed in [5]. This work was later generalized by Tiersten and Mindlin [6] and Mindlin [7] where two-dimensional equations for a piezoelectric plate were systematically derived using power series expansions for the mechanical and electric displacements. Bugdayci and Bogy [8] and Lee et al. [9,10] used trigonometric series expansions for piezoelectric plates, which provides an alternative approach for analyzing plate vibrations more suitable for high-frequency modes. More recently developed plate theories using various sorts of series expansions of the displacements and the electric potential for both single and laminated piezoelectric plates can be found in [11–24]. These expansions are either using a few power series terms, or written on a general higher order fashion that may be used to render solutions that converge to the three dimensional solutions. Exact three dimensional analyses for single and laminated piezoelectric plates having mixed (simply supported) boundary conditions are treated in [25–28]. Numerical methods such as finite element analysis (FEA) has been adopted on both classical and higher order series expansion theories [29–34].

Recently Mauritsson et al. [35] have derived plate equations for a homogenous fully anisotropic elastic plate, using a systematic power series expansion approach, previously adopted for isotropic rods, beams, shells and plates [36–40]. The same general approach has been applied to various piezoelectric layer configurations [41–43]. In the present paper the work in [35] is extended to cover anisotropic piezoelectric plates. The method aims at systematically develop a hierarchy of equations for general piezoelectric plates using power series expansions in the thickness coordinate of the displacement components and the electric potential. Insertion of these expansions into the three dimensional equations of motion leads to recursion relations among the expansion functions, which can be used to eliminate all but some of the lowest order expansion functions. Hereby all fields can be expressed in a finite number of expansions functions without performing any truncations. The power series expansions are subsequently inserted into the three dimensional boundary conditions at the upper and the lower surfaces of the plate. These boundary conditions represent a set of eight scalar equations of motion, including eight unknown

* Corresponding author. Tel.: +46 31 7721521; fax: +46 31 772 3827.
E-mail addresses: karl.mauritsson@his.se (K. Mauritsson), peter.folkow@chalmers.se (P.D. Folkow).
[1] Present address: School of Technology and Society, University of Skövde, SE-541 28 Skövde, Sweden.

http://dx.doi.org/10.1016/j.compstruc.2015.02.023
0045-7949/© 2015 Elsevier Ltd. All rights reserved.

expansion functions, which constitute the system of plate equations. Using variational calculus, the pertinent edge boundary conditions for rectangular plates are obtained in an equally systematic manner. This hierarchy of piezoelectric plate equations can be truncated to any order in the thickness where each studied truncation order is asymptotically correct in line with [35,37,38,44].

The present approach generally differs in several respects from the cited works using power series expansion on piezoelectric plates. The main issues concern the derivation of exact recursion relations where only the lowest order expansion terms need to be considered. Another object is the procedure when collecting terms for the truncation process, resulting in variationally consistent equation systems that are asymptotically correct. It should also be noticed that the present equations are not confined to the static case. Moreover, the plate configuration may be of arbitrary anisotropy without any symmetry classes. One advantage with such a general analysis is that all other cases can be obtained as special cases. The previously derived plate equations for the fully anisotropic elastic case [35] can also be obtained as a special case. As a fully anisotropic, piezoelectric material is described by 21 independent stiffness constants, 18 independent piezoelectric coupling constants and six independent dielectric constants, the explicit expressions for the coefficients in the plate equations become very complicated. For this reason the plate equations for the most general case are derived in a very compact form as four matrix equations, including matrix operators which are recursively defined using the commercial code Mathematica.[2] Hereby it is straightforward to study all types of anisotropy configurations.

As the material configuration for a fully anisotropic material results in complicated expressions, it is natural to study simpler cases of orthotropic plates more in detail. Here the eight plate equations can be added and subtracted in pairs to obtain two uncoupled systems of equations, each of them including four equations and four unknowns. The two uncoupled systems correspond to the symmetric (in-plane) and antisymmetric (out-of-plane) part of the motion, respectively. These equations, including the edge boundary conditions, are explicitly given for the lower truncation orders. To validate the present plate equations results for dispersion curves, eigenfrequencies as well as potential, displacement and stress distribution curves are presented. Both single and laminated plates are studied and comparisons are made to other approximate theories as well as the exact three dimensional theory. The results illustrate both the benchmark property of the higher order truncations and the efficiency of the lower order engineering equations.

2. Problem formulation

Consider a homogeneous piezoelectric plate of thickness $2h$ according to Fig. 1. The material is fully anisotropic with density ρ. The basic equations governing the motion in a piezoelectric continuum are written with abbreviated subscripts [45] as

$$\nabla_{iJ} T_J = \rho \partial_t^2 u_i, \tag{2.1}$$

$$\nabla_i D_i = 0. \tag{2.2}$$

Here vector subscripts are expressed through lower case letters $i = x, y, z$, while abbreviated subscripts are expressed through upper case letters $I = 1, 2, 3, 4, 5, 6$. The mechanical stress, mechanical displacement and electric displacement column matrices are defined through

$$[T_I] = \left(T_{xx} T_{yy} T_{zz} T_{yz} T_{xz} T_{xy}\right)^T, \quad [u_i] = \left(u_x u_y u_z\right)^T, \quad [D_i] = \left(D_x D_y D_z\right)^T. \tag{2.3}$$

[2] Registered trademark of Wolfram Research Inc.

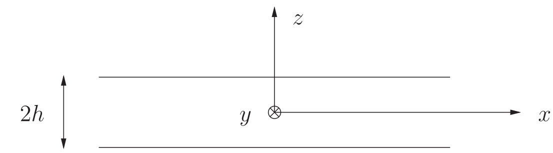

Fig. 1. The geometry.

The divergence vector ∇_i is defined in the usual way, while the divergence stress operator ∇_{iJ} is represented in matrix form through

$$[\nabla_{iJ}] = \begin{pmatrix} \partial_x & 0 & 0 & 0 & \partial_z & \partial_y \\ 0 & \partial_y & 0 & \partial_z & 0 & \partial_x \\ 0 & 0 & \partial_z & \partial_y & \partial_x & 0 \end{pmatrix}, \quad [\nabla_{Ij}] = [\nabla_{iJ}]^T. \tag{2.4}$$

Partial derivatives are expressed as $\partial_x = \partial/\partial_x$ and so on.

For a linear elastic, piezoelectric material the constitutive equations that express the mechanical stresses and the electric displacements in terms of the mechanical displacements and the electric potential, are

$$T_I = c_{IJ} \nabla_{Jk} u_k + e_{Ij} \nabla_j \Phi, \tag{2.5}$$

$$D_i = e_{iJ} \nabla_{Jk} u_k - \epsilon_{ij} \nabla_j \Phi. \tag{2.6}$$

Here the quasistatic approximation is applied, i.e. the electric field is given as the gradient of the electric potential

$$E_i = -\nabla_i \Phi. \tag{2.7}$$

The various material parameters appearing in (2.5) and (2.6) are the 21 independent stiffness constants collected in the symmetric 6×6 matrix

$$[c_{IJ}] = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{12} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{14} & c_{24} & c_{34} & c_{44} & c_{45} & c_{46} \\ c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{56} \\ c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66} \end{pmatrix}, \tag{2.8}$$

the 6 independent dielectric constants collected in the symmetric 3×3 matrix

$$[\epsilon_{ij}] = \begin{pmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{xy} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{xz} & \epsilon_{yz} & \epsilon_{zz} \end{pmatrix}, \tag{2.9}$$

and the 18 independent piezoelectric coupling constants collected in the 3×6 piezoelectric coupling matrix

$$[e_{iJ}] = \begin{pmatrix} e_{x1} & e_{x2} & e_{x3} & e_{x4} & e_{x5} & e_{x6} \\ e_{y1} & e_{y2} & e_{y3} & e_{y4} & e_{y5} & e_{y6} \\ e_{z1} & e_{z2} & e_{z3} & e_{z4} & e_{z5} & e_{z6} \end{pmatrix}, \quad [e_{Ij}] = [e_{iJ}]^T. \tag{2.10}$$

Insertion of (2.5) and (2.6) into (2.1) and (2.2) gives the governing equations for the displacements and the electric potential

$$\nabla_{iJ} c_{JK} \nabla_{Kl} u_l + \nabla_{iJ} e_{Jk} \nabla_k \Phi = \rho \partial_t^2 u_i, \tag{2.11}$$

$$\nabla_i e_{iJ} \nabla_{Jk} u_k - \nabla_i \epsilon_{ij} \nabla_j \Phi = 0. \tag{2.12}$$

3. Power series expansions

To derive plate equations, the displacement components and the electric potential are expanded in power series in the thickness coordinate

$$u_i(x, y, z, t) = \sum_{n=0}^{\infty} z^n u_i^{(n)}(x, y, t), \tag{3.1}$$