Computers and Structures 89 (2011) 1050–1058

Contents lists available at ScienceDirect

Computers and Structures

journal homepage: www.elsevier.com/locate/compstruc

Collapse load evaluation of structures with frictional contact supports under combined stresses

S. Tangaramvong *, F. Tin-Loi

School of Civil and Environmental Engineering, The University of New South Wales, Sydney, NSW 2052, Australia

ARTICLE INFO

Article history:
Received 22 May 2010
Accepted 20 December 2010
Available online 26 January 2011

Keywords:
Combined stresses
Complementarity
Coulomb friction
Limit analysis
Mixed finite element
Unilateral contact

ABSTRACT

This paper extends classical limit analysis to structures for which some supports are subjected to "nonstandard" unilateral frictional contact with the ground. A typical and commonly adopted model is nonassociative Coulomb friction. For such cases, the use of the classical bound theorems is not possible. Moreover, simply solving the governing equations as a mixed complementarity problem (MCP) does not guarantee that the best bound has been calculated. We have therefore developed an approach that attempts to compute, in a single step, the critical (least) upper bound solution by formulating and solving an instance of the challenging class of optimization problems, known as a mathematical program with equilibrium constraints (MPEC). Two examples are provided to illustrate application of the proposed scheme, as well as to highlight some key features of such structures.

1. Introduction

As is well-known, classical limit analysis is a "direct" or "simplified" approach that avoids a computationally expensive time-stepping analysis. Its distinctive feature is the determination, in a single step, of the load factor (or more precisely, its upper and/or lower bounds) at which a critical event (namely plastic collapse) occurs. The upper and lower bound theorems underpinning the classical approach are, however, strictly only applicable to structures that satisfy some rather restrictive requirements, the main ones being conditions of perfect plasticity (e.g. no hardening or softening, normality and convexity of yield surface). Extending such theorems to include some or all "nonstandard" properties, as they are often referred to, is of theoretical and practical value.

The limit analysis of discrete structures continues to be of research interest, see e.g. [1–3]. This has been motivated by three main objectives: (a) application to specific, possibly "nonstandard" engineering situations, (b) overcoming the volumetric locking behavior encountered in plane strain and 3D problems, and (c) development of efficient computational strategies for solving practically motivated problems.

Due to the computational challenges it poses, and at the same time its practical usefulness, the limit analysis of structures involving "nonstandard" unilateral frictional contact is of particular interest. For the commonly assumed Coulomb model the frictional conditions are nonassociative, since sliding is not accompanied by dilation (separation of contact interfaces). Normality with respect to the friction cone is thus no longer applicable, and the classical dual pair of bound theorems is invalid. This problem has been tackled with partial success for frames [4] via the well-known bipotential method [5,6]. We have recently revisited this problem [7], where the best upper bound to the collapse load of such a frame, for which plasticity is caused by bending only, has been successfully computed.

The present paper is an extension to that work. Our aim is still to compute in a single step the best upper bound to the collapse load of a rigid perfectly-plastic structure for which some or all of its supports are in unilateral frictional contact with ground. Monotonically applied loads are assumed, as well as a small deformation regime.

More specifically, the two key extensions to our previous work [7] are as follows. Firstly, the influence of combined stresses in plasticity is included. Moreover, we avoid piecewise linearizing the nonlinear yield surface [8], since this can lead to inaccuracies and introduces a larger, often prohibitive, number of variables. In spite of this nonlinearity, our proposed algorithm appears to have, from the numerical tests performed, the capability to process successfully the problem. Secondly, we extend our approach to more sophisticated finite element models, such as those constructed from the plane strain mixed finite element developed by Capsoni and Corradi [9,10]. This element advantageously offers a locking free capability in incompressibility as well as coarse mesh accuracy. A number of examples, two of which are reported herein, have been successfully solved. The best upper bounds have thus been computed and the influence of frictional coefficient can be

* Corresponding author. Tel.: +61 2 9385 5474; fax: +61 2 9385 6139.
E-mail address: sawekchai@unsw.edu.au (S. Tangaramvong).

0045-7949/$ - see front matter
doi:10.1016/j.compstruc.2010.12.015

assessed. A comparison of associative against nonassociative friction laws can also be made.

The generic idea underpinning our approach is to formulate, within a mathematical programming framework, the extended limit analysis by simply collecting the governing relations (static, kinematic and material constitution) for the analysis, augmented by the frictional contact conditions. This leads to a special mathematical programming problem known as a mixed complementarity problem (MCP) [11]. For the nonassociative friction model, the MCP, however, may admit multiple collapse load solutions [7]. We attempt to directly capture the critical (least) upper bound solution by formulating and solving the proposed limit analysis as a nonconvex and nonsmooth optimization problem known in the literature as a mathematical program with equilibrium constraints (MPEC) [12]. The feature (and difficulty) of an MPEC lies in the presence of disjunctive complementarity constraints.

The organization of this paper is as follows. In the next Section 2, we review the basic conditions for the adopted unilateral frictional contact model. Section 3 describes our discrete finite element model and the governing relations. In Section 4, the extended limit analysis is formulated first as an MCP. We show that the necessary skew-symmetric conditions, necessary for the existence of a dual pair of bound theorems, can be recovered when piecewise linearization and associative friction are assumed. The MPEC formulation is then presented together with a mention of some computational difficulties that can arise for this class of problems. We outline in Section 5 three nonlinear programming (NLP)-based solution algorithms that we have used to solve MPECs. In brief, the MPEC is processed as a series of iterative NLP subproblems after suitably "treating" the complementarity constraints. Two numerical examples are provided in Section 6. The first example involves a multistory frame modeled using bar elements [13], whilst the second considers a plane strain structure obeying von Mises yield criteria modeled using mixed finite elements [9,10]. Finally, some pertinent concluding remarks are drawn in Section 7.

A word regarding notation is in order. Vectors and matrices are indicated in bold. A real vector $\mathbf{x}$ of size m is indicated by $\mathbf{x} \in \mathcal{R}^m$ and a real $m \times n$ matrix $\mathbf{A}$ by $\mathbf{A} \in \mathcal{R}^{m\times n}$. For brevity, a vector of functions $\mathbf{f}(\mathbf{x}) : \mathcal{R}^m \to \mathcal{R}^n$ is written simply as $\mathbf{f} \in \mathcal{R}^n$.

2. Basic contact conditions

The basic contact conditions [7,13] that apply at any unilateral frictional contact point k at an inclination β are reviewed. This can be briefly described with reference to Fig. 1, where subscripts n and t denote respectively the normal and tangential directions to the interface. A general friction law is shown in Fig. 1b, where ϕ define the friction angle ($\tan\phi$ represents the traditional coefficient of friction μ), φ is the dilatancy angle ($\varphi = 0$ for Coulomb friction), (r_n^k, r_t^k) are interface forces and $(\dot{u}_n^k, \dot{u}_t^k)$ are the corresponding

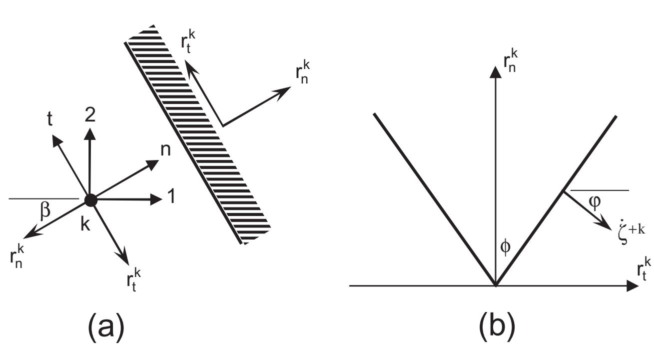

Fig. 1. Generic frictional contact k (a) forces, (b) general friction law.

displacement rates at the contact point. The mechanical behavior at this generic contact node can be considered separately in the normal and in the tangential directions at the interface. This is detailed in the following. All contacts are also assumed to have a zero initial gap with the fixed ground.

In the first instance, sliding in either of two opposite directions of the contact k is modeled through the following complementarity conditions:

$$\pi_c^{+k} = r_n^k \sin\phi - r_t^k \cos\phi \geqslant 0, \quad \dot{\zeta}^{+k} \geqslant 0, \quad \pi_c^{+k}\dot{\zeta}^{+k} = 0, \tag{1}$$

$$\pi_c^{-k} = r_n^k \sin\phi + r_t^k \cos\phi \geqslant 0, \quad \dot{\zeta}^{-k} \geqslant 0, \quad \pi_c^{-k}\dot{\zeta}^{-k} = 0, \tag{2}$$

where variables $\dot{\zeta}^{+k}$ and $\dot{\zeta}^{-k}$ are sliding multiplier rates (named in analogy with plastic multiplier rates used in plasticity). In essence, similar to yield conditions in plasticity (see e.g. [8]), relations (1) and (2) indicate that sliding can only occur if limiting friction has been reached.

The nonpenetration (Signorini) condition is governed by the following obvious complementarity condition, expressed in rate form:

$$\pi_n^k = -\dot{u}_n^k \geqslant 0, \quad r_n^k \geqslant 0, \quad \pi_n^k r_n^k = 0. \tag{3}$$

Clearly, (3) describes the fact that if $-\dot{u}_n^k > 0$, then $r_n^k = 0$, and that if $-\dot{u}_n^k = 0$, then $r_n^k \geqslant 0$. Physically, this relation embodies the requirement that nodes cannot penetrate rigid obstacles.

3. Generic finite element model

A suitably space discretized rigid perfectly-plastic structural system for which some of the supports may be subjected to frictional contact condition of the afore-mentioned type is considered. It is assumed that the structure under consideration has been discretized as an aggregate of finite elements. The material behavior is directly reflected by the elemental behavior. More explicitly, we refer to the class of finite elements expressed in intrinsic, natural (in Prager's generalized sense) variables [8]. This implies that the scalar product of generalized stress $\mathbf{Q}^i$ and plastic strain rate $\dot{\mathbf{p}}^i$ vectors represents virtual work in the element i concerned and is invariant with respect to rigid body motion. Two obvious examples of this element class are bars (trusses and lumped compliance frame elements) and constant strain, homogeneous two- and three-dimensional elements (3-node plane stress triangles and 4-node tetrahedrons, respectively).

The external loads are proportionally applied, through a single load multiplier α, to the nodes. Distributed loads are simulated as equivalent concentrated forces applied on an appropriate number of nodes. The unconstrained nodal forces $\mathbf{F}^i$, defined with respect to a global reference axis system, are then expressed in terms of the load multiplier α and the given basic nodal load vector $\mathbf{f}^i$ as $\mathbf{F}^i = \alpha\mathbf{f}^i$.

Within the assumed small deformation regime, both equilibrium and compatibility relations are linear. In particular, equilibrium between the nodal applied forces $\alpha\mathbf{f}^i$, the interface forces (r_n^k, r_t^k) and the elemental stress resultants $\mathbf{Q}^i$ can be described as

$$\mathbf{C}^{i\mathrm{T}}\mathbf{Q}^i = \alpha\mathbf{f}^i - \mathbf{C}_n^{k\mathrm{T}} r_n^k - \mathbf{C}_t^{k\mathrm{T}} r_t^k, \tag{4}$$

where $\mathbf{C}^i$ is an elemental linear compatibility matrix, and $\mathbf{C}_n^k$ and $\mathbf{C}_t^k$ are compatibility matrices pertaining to frictional support k in the normal and tangential directions, respectively. The compatibility condition between the nodal displacement rates $\dot{\mathbf{u}}^i$ and the plastic strain rates $\dot{\mathbf{p}}^i$ is given by

$$\dot{\mathbf{p}}^i = \mathbf{C}^i\dot{\mathbf{u}}^i. \tag{5}$$

As already mentioned, the constitutive law adopted is based on a rigid perfectly-plastic material assumption. The relations