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ORIGINAL ARTICLE

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On Oseen flows for large Reynolds numbers

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Abstract This investigation offers a detailed analysis of solutions to the two-dimensional Oseen problem in the exterior of an obstacle for large Reynolds numbers. It is motivated by mathematical results highlighting the important role played by the Oseen flows in characterizing the asymptotic structure of steady solutions to the Navier–Stokes problem at large distances from the obstacle. We compute solutions of the Oseen problem based on the series representation discovered by Tomotika and Aoi (Q J Mech Appl Math 3:140–161, 1950) where the expansion coefficients are determined numerically. Since the resulting algebraic problem suffers from very poor conditioning, the solution process involves the use of very high arithmetic precision. The effect of different numerical parameters on the accuracy of the computed solutions is studied in detail. While the corresponding inviscid problem admits many different solutions, we show that the inviscid flow proposed by Stewartson (Philos Mag 1:345–354, 1956) is the limit that the viscous Oseen flows converge to as $Re \to \infty$. We also draw some comparisons with the steady Navier–Stokes flows for large Reynolds numbers.

Keywords Oseen equation · External flows · Wakes · Steady flows · Computational methods

1 Introduction

This research concerns steady incompressible flows past obstacles in unbounded domains, and we investigate solutions of Oseen's approximation to the Navier–Stokes system in the limit of vanishing viscosities. Motivated by certain mathematical results highlighting the role of the Oseen problem for some open questions in theoretical hydrodynamics, we will attempt to provide a modern look at this classical problem. We consider two-dimensional (2D) flows past a circular cylinder A with the unit radius in an unbounded domain $\Omega \triangleq \mathbb{R}^2 \backslash \overline{A}$ ("$\triangleq$" means "equal to by definition"). It is assumed that the flow is generated by the free stream $\mathbf{u}_\infty = 1\,\mathbf{e}_x$ at infinity, where $\mathbf{e}_x$ is the unit vector associated with the OX axis. Denoting $\mathbf{u} = [u, v]^T$ the velocity field, p the pressure, and assuming the fluid density is equal to unity, the Oseen system takes the following form

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$$\mathbf{u}_\infty \cdot \nabla \mathbf{u} + \nabla p - \frac{1}{Re}\Delta \mathbf{u} = \mathbf{0} \quad \text{in } \Omega, \tag{1a}$$
$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega, \tag{1b}$$
$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial A, \tag{1c}$$
$$\mathbf{u} \to \mathbf{u}_\infty \quad \text{as } |\mathbf{x}| \to \infty, \tag{1d}$$

where $\mathbf{x} = [x, y]^T$ is the position vector, and Re denotes the Reynolds number. The coordinate system is fixed at the obstacle, and the no-slip boundary conditions were assumed at the obstacle boundary ∂A. Noting that the obstacle diameter is $d = 2$, the Reynolds number is calculated as

$$Re = \frac{2|\mathbf{u}_\infty|}{\mu}, \tag{2}$$

where μ is the dynamic viscosity. System (1) is a linearization of the Navier–Stokes problem obtained by replacing the advection velocity in the nonlinear term with the free stream $\mathbf{u}_\infty$. It therefore represents an alternative model to the Stokes approximation in which the nonlinear term is eliminated entirely, resulting in a problem which admits no solutions in 2D, a fact known as Stokes' paradox [1]. While the time-dependent generalizations were recently considered [2], Oseen's linearization, as well as Stokes', is typically studied in the time-independent setting.

In the 1950s and 1960s, the Oseen equation generated significant interest, since being analytically tractable, its solutions offered valuable quantitative insights into properties (such as, e.g., drag) of low Reynolds number flows past bluff bodies, otherwise unavailable in the pre-CFD era. Different solutions of the Oseen equation, either in 2D or in 3D, were constructed by Lamb [3], Oseen himself [4], Burgess [5], Faxén [6], and Goldstein [7] to mention the first attempts only. It should be, however, emphasized that these early "solutions" satisfied governing equation (1a), or boundary condition (1c), or both, in an approximate sense only. The first solution (in the 2D case for the flow past a circular cylinder) satisfying exactly both the governing equation and the boundary conditions, albeit depending on an infinite number of constants, is that of Faxén [6]. It was later rederived in simpler terms by Tomotika and Aoi [8], whereas Dennis and Kocabiyik [9] obtained analogous solutions for flows past elliptic cylinders inclined with respect to the oncoming flow. We also mention the work of Chadwick et al. [10,11] who considered solutions expressed in terms of convolution integrals with "Oseenlets", i.e., Green's functions for Eq. (1). Even a cursory survey of these results and the different applications that the Oseen equation found in the studies of wake flows, both fundamental and applied, is beyond the scope of this paper. Instead, we refer the reader to the excellent historical reviews by Lindgren [12] and Veysey II and Goldenfeld [13], and the monograph by Berger [14] for a survey of historical developments and different applications. Following the advent of computational fluid dynamics which made it possible to solve the complete Navier–Stokes system numerically, the interest in the Oseen approximation as a means of studying real flows subsided. We remark that the Oseen system is still occasionally used as a test bed for validating different computational approaches (e.g., stabilization of finite element discretizations [15], or artificial boundary conditions on truncated computational domains [16]).

The present investigation reflects a renewed interest in the Oseen system which comes from a somewhat different direction. It is motivated by certain results of the mathematical analysis concerning the far-wake structure in the steady-state solutions of the Navier–Stokes equation in unbounded domains. The related question of the asymptotic, as $Re \to \infty$, structure of the separated 2D flow past a bluff body was studied extensively using methods of the asymptotic analysis since the 1960s. A number of different limiting solutions have been proposed which fall into two main categories illustrated schematically in Fig. 1, namely, flows characterized by slender wakes extending to infinity such as the Kirchhoff-type solutions with free streamlines [17,18], and flows with wide closed wakes reminiscent of the Prandtl–Batchelor limiting solution [19] with two counter-rotating vortices attached to the obstacle. However, the only asymptotic solution which has been found to be self-consistent and which has withstood the test of time is that of Chernyshenko [20], see also [21]. It features two large recirculation regions with the width and length growing proportionally to Re and the drag coefficient decreasing to zero with Re. This type of asymptotic solution was later generalized in different directions, namely, for flows past rows of obstacles in [22] and for flows with stratification in [23]. As concerns computational studies, the work of Fornberg [24,25] was pioneering and still represents the state of the art in this area. More recently, it was followed by computational studies of steady flows past rows of obstacles in [26,27] and flows past obstacles in channels [28], whereas steady three-dimensional (3D) flows past a sphere were studied in [29]. The results of Fornberg [25] for Reynolds numbers up to about 300 feature a slender elongated wake bubble, whereas for higher values of the Reynolds number, these solutions develop a significantly wider wake