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Generalized stochastic finite element method in elastic stability problems

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ABSTRACT

The main issue of this paper is the stability analysis of elastic systems with random parameters using the Generalized Stochastic Finite Element Method. The Taylor expansion with random coefficients of nth order is used to express all random functions and to determine up to fourth order probabilistic moments of the critical force or critical pressure. The response function method assists to determine higher order partial derivatives of the structural response instead of the Direct Differentiation Method employed widely before. This approach is examined on the classical Euler problem, 2D and 3D steel frames as well as in addition to the cylindrical shell with some geometrical parameters defined as the Gaussian variables. The comparison of the GSFEM *versus* the Monte-Carlo simulation on the Euler problem proves the probabilistic convergence of this new technique.

1. Introduction

Stochastic or probabilistic buckling of the engineering structures is still an important research area since many of these structures exhibit unpredictable fluctuations of the material and geometrical parameters [1–3]; they can also be subjected to dynamic loadings [4,5]. These variations, sometimes of apparently local character [6,7], may significantly change the critical force as one may suppose on the basis of the Euler formula. The stability theory has still many open questions, even in the area of simple prismatic or thin-walled elastic beams, so that probabilistic analytical approaches are not of a general character and, furthermore, are not available for more complex problems like buckling of plates and/or shells with no trivial boundary conditions. The second reason why to deal with this problem is that the critical behavior is decisive for many structures which need to be optimally designed and, as it is documented by the engineering practice, is frequently not accounted for. Since the new Eurocodes for engineering design introduce the necessity of probabilistic design and reliability index determination, there is no doubt that this research area may be important in at least civil engineering.

There are several numerical methods to analyze the buckling phenomenon in the engineering systems with random parameters. One may find a variety of works concerning the stability of structures with random parameters - besides the simulation techniques such as crude or weighted Monte-Carlo scheme (MCS), the Metropolis approaches, the spectral methods as well as hybrid semi-analytical approaches one may find the entire family of perturbation methods [8,9]. The main value of the last class for the engineers is that the computational time is relatively short, there is no need to use massive computers and that they are relatively easily implementable into any computer system. The quality of the stochastic perturbation method (strongly) depends to a high degree on the user's programming skills to include as many higher order terms as it is possible.

In the view of above, the main aim of this paper is to show the application of the generalized nth order perturbation method implemented into the academic FEM software to find the probabilistic moments for several engineering case studies: (a) the Euler critical force, (b) single-aisle 2D and 3D steel frame model, (c) multi-aisle steel frame and (d) polymer underground semi-cylindrical shell. The main goals here are (a) to validate the perturbation method against the MCS, (b) to compare 2D and 3D probabilistic models, (c) to distinguish between various orders perturbation analyses and various orders response polynomials and finally (d) to provide large scale FEM analysis using the RFM technique. The integral part of the computational tools employed in this work is the computer algebra system MAPLE, which is used for both simulation purposes as well as for the processing of the probabilistic characteristics and automatic differentiation in the stochastic perturbation approach. This technique does not require straightforward differentiation of equilibrium equations and their further solution but enables a direct determination of analytical polynomial interrelation between the chosen structural response and given random input. This approach is of a special value because of its wide applicability in nonlinear problems of modern computational mechanics and engineering. Now, the nonlinear weighted least squares fitting technique is used instead of the classical Newton method to determine and to smoothen all the structural response functions thanks to the

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application of symbolic calculus. Let us finally note that all the numerical tests included in the paper deal with the Gaussian random variables but having implemented all equations in symbolic environment one may easily replace this distribution with some non-Gaussian quantities [10] such as the lognormal or the Weibull one, for instance. The main difficulty in this case would be practical determination of its basic parameters and the results interpretation, especially in engineering applications.

2. The generalized stochastic finite element method basis

Let us introduce the random variable $b \equiv b(\omega)$ and its probability density function as $p(b)$. Then, the expected values as well as its central mth probabilistic moments are defined as

$$E[b] \equiv b^0 = \int_{-\infty}^{+\infty} bp(b)db \tag{1}$$

and

$$\mu_m(b) = \int_{-\infty}^{+\infty} (b - E[b])^m p(b)db. \tag{2}$$

The basic idea of this stochastic perturbation approach is to expand all the input variables and all the state functions of the considered problem via Taylor series about the additional expected values using the parameter $\varepsilon > 0$. In the case of random critical force P_{cr} depending on some random input quantity b, the following expression is employed:

$$P_{cr} = P_{cr}^0 + \sum_{n=1}^{\infty} \frac{1}{n!} \varepsilon^n \frac{\partial^n P_{cr}}{\partial b^n} (\Delta b)^n, \tag{3}$$

where

$$\varepsilon \Delta b = \varepsilon (b - b^0) \tag{4}$$

is the first variation of b around its expected value b^0. We will derive the expected value for the critical force P_{cr} in the view of above expansion as

$$E[P_{cr}] = \int_{-\infty}^{+\infty} P_{cr}(b)p(b)db = \int_{-\infty}^{+\infty} \left(P_{cr}^0 + \sum_{n=1}^{\infty} \frac{1}{n!} \varepsilon^n \frac{\partial^n P_{cr}}{\partial b^n} \Delta b^n \right) p(b)db. \tag{5}$$

Let us remind that this power expansion is valid only if the state function is analytic in ε, the series converge and, therefore, any criteria of convergence should include the magnitude of the perturbation parameter; perturbation parameter is usually taken simply as equal to 1 in engineering computations. Contrary to most previous analyses in this area, now the quantity ε is treated as the expansion parameter in further analysis, so that it is included explicitly in all the further derivations demanding analytical expressions.

From the numerical point of view, the expansion provided by the formula (3) is carried out for the summation over the finite number of components, whereas the integral given in definition (5) is never calculated with infinite limits – usually it has lower and upper bounds driven by physical meaning of the specific parameter or just the experimental works. Having Gaussian input in the form of $b(\omega)$ or another symmetric probability distribution function one can show that

$$E[P_{cr}] = P_{cr}(b^0) + \frac{1}{2}\varepsilon^2 \frac{\partial^2 P_{cr}(b^0)}{\partial b^2} \mu_2(b^0) + \frac{1}{2m!}\varepsilon^m \frac{\partial^{2m} P_{cr}(b^0)}{\partial b^{2m}} \mu_{2m}(b^0) + \cdots \tag{6}$$

This expected value can be calculated analytically or symbolically computed only if it is given as some analytical function of the random input parameter b; many existing models in various branches of civil engineering can be adopted to achieve this goal [11,12]. Computational implementation of the symbolic calculus programs (with automatic partial differentiation of even complex real functions), combined with powerful visualization of probabilistic output moments, ensures the fastest solution of such problems. Further, thanks to such a series representation of the random output, any desired efficiency of the expected values as well as higher probabilistic moments can be achieved by an appropriate choice of the expansion length and some additional correction available in the parameter ε, which depend on the input probability density function (PDF) type, interrelations between the probabilistic moments, acceptable error of the computations etc. This choice can be made by comparative studies with sufficiently long (almost infinite) series of Monte-Carlo simulations or theoretical results obtained from the direct symbolic integration. Similar considerations lead to the 6th order expressions for a variance; there holds [13,14]

$$\begin{aligned} Var(P_{cr}(b)) = {} & \varepsilon^2 \mu_2(b) \left(\frac{\partial P_{cr}(b^0)}{\partial b} \right)^2 \\ & + \varepsilon^4 \mu_4(b) \left(\frac{1}{4} \left(\frac{\partial^2 P_{cr}(b^0)}{\partial b^2} \right)^2 + \frac{2}{3!} \frac{\partial P_{cr}(b^0)}{\partial b} \frac{\partial^3 P_{cr}(b^0)}{\partial b^3} \right) \\ & + \varepsilon^6 \mu_6(b) \left(\left(\frac{1}{3!} \right)^2 \left(\frac{\partial^3 P_{cr}(b^0)}{\partial b^3} \right)^2 + \frac{1}{4!} \frac{\partial^2 P_{cr}(b^0)}{\partial b^2} \frac{\partial^4 P_{cr}(b^0)}{\partial b^4} \right. \\ & \left. + \frac{2}{5!} \frac{\partial P_{cr}(b^0)}{\partial b} \frac{\partial^5 P_{cr}(b^0)}{\partial b^5} \right). \end{aligned} \tag{7}$$

Quite similarly, using the lowest order expansions, it is possible to derive third central probabilistic moments as

$$\begin{aligned} \mu_3(P_{cr}(b^0)) &= \int_{-\infty}^{+\infty} (P_{cr}(b) - E[P_{cr}(b)])^3 p(b)db \\ &\cong \frac{3}{2}\varepsilon^4 \mu_4(b) \left(\frac{\partial P_{cr}(b^0)}{\partial b} \right)^2 \frac{\partial^2 P_{cr}(b^0)}{\partial b^2} \\ &\quad + \frac{1}{8}\varepsilon^6 \mu_6(b) \left(\frac{\partial^2 P_{cr}(b^0)}{\partial b^2} \right)^3, \end{aligned} \tag{8}$$

as well as fourth central probabilistic moment in the form of

$$\begin{aligned} \mu_4(P_{cr}(b^0)) &= \int_{-\infty}^{+\infty} (P_{cr}(b) - E[P_{cr}(b)])^4 p(b)db \\ &\cong \varepsilon^4 \mu_4(b) \left(\frac{\partial P_{cr}(b^0)}{\partial b} \right)^4 \\ &\quad + \frac{3}{2}\varepsilon^6 \mu_6(b) \left(\frac{\partial P_{cr}(b^0)}{\partial b} \frac{\partial^2 P_{cr}(b^0)}{\partial b^2} \right)^2 \\ &\quad + \frac{1}{16}\varepsilon^8 \mu_8(b) \left(\frac{\partial^2 P_{cr}(b^0)}{\partial b^2} \right)^4. \end{aligned} \tag{9}$$

Let us mention that it is necessary to insert the relevant central probabilistic moments of the input random variable in each of those equations to get the algebraic form convenient for symbolic computations. A recursive derivation of the particular perturbation order equilibrium equations can be powerful in conjunction with symbolic packages with automatic differentiation tools only; it can potentially extend the area of stochastic perturbation technique applications in computational physics and engineering outside the random processes with small dispersion about their expected values. Hence, there is no need to implement directly exact