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Existence and computation of Berge equilibrium and of two refinements[☆]

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ABSTRACT

This paper presents existence conditions as well as computation methods for Berge equilibrium and two refinements: Berge–Vaisman equilibrium and Berge–Nash equilibrium. Each equilibrium concept is interpreted and illustrated on the basis of relevant examples and general existence conditions satisfying weak continuity and quasi-concavity conditions are provided.

1. Introduction

Berge equilibrium (BE) (Berge, 1957; Zhukovskii and Chikrii, 1994) and its refinements (Vaisman, 1994, 1995; Abalo and Kostreva, 1996, 2004) have attracted increasing attention recently in the game theoretic literature for its theoretical contributions to the modeling of social norms and prosocial behaviors. Colman et al. (2011) define and interpret BE as a mutual support equilibrium. In playing BE, agents support each other by mutually selecting actions that maximize the welfare of others. The assumption is that they adopt such behavioral norms because of the reciprocal dimension implied. Courtois et al. (2015) develop a situational theory in which players follow either a Berge or Nash behavior rule. The hypothesis is that agents adopt the behavior rule that is the most beneficial to them. This means that in social situations such as trust games or social dilemmas, players would mutually support each others, while in competitive situations such as zero sum games, they would simply maximize their welfare independently from the welfare of others.

A key criticism of this theory is that BE is not immune to unilateral deviation. It is not either necessarily compatible with individual rationality. Two refinements of the concept address partly these issues. The Berge–Vaisman equilibrium (BVE) (Vaisman, 1994, 1995), restricts the set of BE to the subset in which the equilibrium gain of each agent is no lower than his *maximin* payoff. BVE precludes all BE that are not individually rational (see Crettez (2016) for sufficient conditions for BE to be BVE). Another refinement defined by Abalo and Kostreva (1996, 2004) goes further and restricts the set of BE to the subset in which Berge strategy is a best reply. Berge–Nash equilibrium (BNE) combines the properties of Berge and of Nash equilibrium (NE) and admits the advantage of characterizing the set of BE that are incentive compatible.

The current paper offers general existence conditions as well as computation methods for BE (in coalitional or individualistic form), BVE, and BNE. Up until now, existence results have only been proposed for BE in coalitional form (CBE) and for BNE for n-player games in which (1) the strategy spaces are nonempty, convex, and compact, and (2) players have continuous and quasiconcave payoff functions (Nessah et al., 2007; Larbani and Nessah, 2008). Existence results for BVE have also been proposed, but these results are limited to the case of differential games with quadratic payoffs (Vaisman, 1994, 1995; Zhukovskii and Chikrii, 1994). Nessah and Larbani (2014) gave a general existence result that weakens the concavity and continuity conditions for BE in individualistic form (also called Berge–Zhukovskii equilibrium (BZE)) in 2-player games. Generalizing these results, this paper investigates new existence conditions for CBE, BNE, and BVE in n-player games that satisfy weak continuity and quasi-concavity conditions. The

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proposed existence conditions for the three equilibrium concepts are theoretically related and based on the notions of diagonal transfer quasi-concavity and diagonal transfer continuity (Baye et al., 1993). The two notions are weak concepts of quasi-concavity and continuity which adopt a basic idea of transferring a set of strategy profile(s) to another set of strategy profile(s). The key contribution of the paper is therefore to provide simple existence and computation methods conditions that allow for BVE, BNE, BZE and CBE to be used in the largest possible class of games.

The paper is organized as follows. In Section 2, we define CBE, BZE, BVE and BNE and interpret these equilibrium concepts on the basis of several examples. In Section 3, we provide the sufficient conditions for their existence, and we derive operational procedures for their computation Section 4 concludes and makes suggestions for further research concerning this topic.

2. Definitions and interpretations

Consider the game

$$G = (X_i,\ u_i)_{i\in I}, \tag{2.1}$$

where $I = \{1, 2, \ldots, n\}$ is the set of players, $X = \prod_{i\in I} X_i$ is the set of strategy profiles in the game, X_i is the set of strategies of player i, $X_i \subset E_i$, E_i is a vector space, and $u_i : X \longrightarrow \mathbb{R}$ is the bounded payoff function of player i.

Let $\Im$ denote the set of all coalitions (*i.e.*, nonempty subsets of I). For each coalition $C \in \Im$, we have a complementary coalition set denoted by $-C$. If C is reduced to a singleton $\{i\}$, the set $-C$ is denoted by $-i$. We also denote $X_C = \prod_{i\in C} X_i$ as the set of strategies of the players in coalition C. Let $R = \{R_i\}_{i\in M}$ be a partition of the set of players I where $M = \{1, \ldots, s\}$ is an index set. Any strategy profile $x = (x_1, \ldots, x_n) \in X$ can be written $x = (x_{R_1}, x_{R_2}, \ldots, x_{R_s})$.

We start with a definition of CBE as introduced in Berge (1957, p. 88–89).

Definition 2.1 (*Berge, 1957*)**.** Consider the game (2.1) and let $R = \{R_i\}_{i\in M} \subset \Im$ be a partition of I and $S = \{S_i\}_{i\in M}$ be a set of subsets of I. A feasible strategy $\overline{x} \in X$ is an equilibrium point for the set R relative to the set S or a *coalitional Berge equilibrium* (CBE) for (2.1) if

$$u_{r_m}(\overline{x}) \geq u_{r_m}(\overline{x}_{-S_m}, x_{S_m}),$$
$$\text{for each given } m \in M,\ \text{any } r_m \in R_m,\ \text{and } x_{S_m} \in X_{S_m}.$$

A strategy profile $\overline{x}$ is a CBE if no player in any coalition R_m in R, can be better off when the players in the corresponding coalition S_m in S, deviate from their BE strategy profile $\overline{x}_{S_m}$. This means that at CBE, the players in coalition S_m play a strategy profile that maximizes the payoff of the players in coalition R_m, but they neglect or ignore their own payoffs (when $S_m \bigcap R_m = \emptyset$). At CBE the payoffs of the players in S_m are taken care of by other players, making this rule of conduct resemble a reciprocal behavior. This equilibrium concept could be summed up by the following perspective: "I care about your welfare because you care about mine making both of us better off".

Example 2.1 (*Climate Coalitional Game*)**.** Consider an economy consisting of n countries. Let $I = \{1, 2, \ldots, n\}$ be the index set of countries, $R = \{R_1, \ldots, R_s\}$ be a partition of I and $S = \{S_1, \ldots, S_s\}$ be a coalition structure.

Let $e_i \geq 0$ and $q_i = g_i(e_i)$ denote the emission level and the output resulting from this emission level for country i. Let $z(e_1, \ldots, e_n) = \sum_{i\in I} e_i$ be the total pollution level and $v_i(z)$ be country's i disutility resulting from this pollution. The net utility of player i is then

$$u_i(e_1, \ldots, e_n) = g_i(e_i) - v_i(z(e_1, \ldots, e_n)).$$

For each coalition R_m, $m = 1, \ldots, s$, define the utility of any country in this coalition as follows:

$$\widetilde{U}_h(e_1, \ldots, e_n) = \sum_{j\in R_m} u_j(e_1, \ldots, e_n), \quad \text{for each } h \in R_m.$$

Then for any given (R, S), we associate a game $(X_i, \widetilde{U}_i, R, S)_{i\in I}$. Consider $n = 4, g_i(e_i) = \alpha_i\sqrt{e_i}, v_i(z) = \beta_i z + \gamma_i$, for all $i = 1, 2, 3, 4$ and assume that $R_m = S_m$, for each $m = 1, \ldots, s$.

If $R = \{\{1, 2\}, \{3, 4\}\}$, the unique CBE is given by

$$\overline{e}_i = \frac{\alpha_i^2}{4(\beta_1 + \beta_2)^2},\ \ i = 1, 2 \text{ and}$$
$$\overline{e}_i = \frac{\alpha_i^2}{4(\beta_3 + \beta_4)^2},\ \ i = 3, 4.$$

If $R = \{\{1, 2, 3\}, \{4\}\}$, the unique CBE is given by

$$\overline{e}_i = \frac{\alpha_i^2}{4(\beta_1 + \beta_2 + \beta_3)^2},\ \ i = 1, \ldots, 3 \text{ and } \overline{e}_4 = \frac{\alpha_4^2}{4\beta_4^2}.$$

If $R = \{\{1, 2, 3, 4\}\}$, the unique CBE is given by

$$\overline{e}_i = \frac{\alpha_i^2}{4(\beta_1 + \beta_2 + \beta_3 + \beta_4)^2},\ \ i = 1, \ldots, 4. \quad \blacksquare$$

If we let $j \in S_m$, and since the family of coalitions R is a partition of the set of players I, there exists some $p \in M$ such that $j \in R_p$. According to the definition of CBE, the players of the corresponding coalition S_p maximize the payoff functions of the players in R_p, and since $j \in R_p$, the payoff of player j is also maximized by the players of S_p. We deduce that at a CBE, each player maximizes the payoff of at least one of the other players and in turn, his own payoff is maximized by at least one other player. Reformulating this coalitional equilibrium concept from an individualistic perspective, we obtain what we call a BZE as defined by Zhukovskii (1985).

Definition 2.2 (*Zhukovskii, 1985*)**.** A strategy profile $\overline{x} \in X$ is a *Berge–Zhukovskii equilibrium* (BZE) of the game (2.1) if

$$u_i(\overline{x}) \geq u_i(\overline{x}_i, y_{-i}),\ \text{for each, given } i \in I \text{ and } y_{-i} \in X_{-i}. \tag{2.2}$$

In order to show that BZE is a special case of CBE, assume that $M = I, R_i = \{i\}, i \in I, S_i = -i$, and $i \in I$. This means that when playing the BZE strategy $\overline{x}_i$, a player $i \in I$ yields his highest possible utility when other players also play according to their BZE strategy. This also means that this same player, when following strategy $\overline{x}_i$, cannot obtain a maximum payoff unless the remaining players $-i$ are willing to play the strategy $\overline{x}_{-i}$. We deduce that if all players play $\overline{x}$, then all payoffs are maximized but if at least one of the players j deviates from his equilibrium strategy, then the payoff of any player i in $-j$, the resulting strategy profile, is at most equal to his payoff $u_i(\overline{x})$ in the resulting profile.

Example 2.2 (*Costly Contribution Game*)**.** Consider a three players contribution game in which each player can either contribute to a collective action or retract his contribution. Set $I = 3$ as the number of players and let $X_i = [-1, 1]$ be the strategy space of player's i, with $i = 1, 2, 3$. Let us first consider a symmetric payoff function such that:

$$u_i(x) = -x_i + \sum_{j\in -i} x_j,\ \ i = 1, 2, 3. \tag{2.3}$$

We can easily see that $\overline{x} \in X$ is a BZE if and only if $\max_{y_{-i}\in X_{-i}} u_i(\overline{x}_i, y_{-i}) = u_i(\overline{x})$, for any $i = 1, 2, 3$. We deduce that $\overline{x} = (1, 1, 1)$ is the unique pure strategy BZE of this game and $u_i(\overline{x}) = 1$, for all $i = 1, 2, 3$. Notice that by playing BZE, each player ends up better off than by playing Nash equilibrium (NE). The unique