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A one-sided many-to-many matching problem

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ABSTRACT

This study discusses a one-sided many-to-many matching model wherein agents may not be divided into two disjoint sets. Moreover, each agent is allowed to have multiple partnerships in our model. We restrict our attention to the case where the preference of each agent is single-peaked over: (i) the total number of partnerships with all other agents, and (ii) the number of partnerships that the agent has with each of the other agents. We represent a matching as a multigraph, and characterize a matching that is stable and constrained efficient. Finally, we show that any direct mechanism for selecting a stable and constrained efficient matching is not strategy-proof.

1. Introduction

This study considers the following problem. Several pre-season or post-season baseball teams want to play a number of practice games. A game between two teams is realized if there is consent from both teams. We call the set of all games played among the teams *matching*. Each team has its preference over the set of the matchings. That is, each team may be particular about the number of games they play and their opponents. In such a situation, which matching is stable? Which matching is efficient? Is a stable matching efficient? In addition, how should an authority design a matching?

To understand the problem, we consider a preliminary example. There are four baseball teams, 1, 2, 3 and 4. All teams are indifferent on the opponents but team 1 wants to play four times and the other teams want to play twice, because of the policies of the directors of the teams. Then, for example, consider the matching such that the games between 2 and 3, 2 and 4, and 3 and 4 are, respectively, played once. The left graph in Fig. 1 depicts the matching where the nodes represent the teams and an edge between two teams represents one game between them. Then, this matching can be considered stable, because teams 2, 3 and 4 do not want to play anymore, although team 1 does want to play more. However, the matching is not Pareto efficient. For example, consider a new matching such that team 1 plays with 2 twice and with 3 and 4 once, respectively, and 3 and 4 play once. The right graph in Fig. 1 represents the matching. Then, this new matching Pareto dominates the former matching, because 1 prefers the latter to the former and the others are indifferent.

There are several matching models that are related to the model in this study.[1] The marriage model introduced by Gale and Shapley (1962) includes several men and women, and a matching is a set of couples between a man and a woman. The roommates model is also introduced by Gale and Shapley (1962). In the model, they consider a more general situation where the agents may not be divided into two disjoint sets. In our model, a team can play more than one game. The many-to-many matching model introduced by Roth (1984, 1985) is another generalization of the marriage model. A typical real world market to which his model can be applied is a labor market, where a firm can hire a set of workers and a worker can be employed by a set of firms. However, in the many-to-many matching model, the agents are divided into two disjoint sets and an agent can form at most one partnership with another agent. On the other hand, in our model, the teams may not be divided into two disjoint sets and a team can play two or more games against another team. That is, we consider a one-sided matching model that an agent can form multiple partnerships.

Since a matching can be represented by a graph as that in Fig. 1, our model can be regarded as a network formation model. See, for example, Goyal (2007) and Jackson (2008) for the survey of network formation models. Most of the previous studies restrict their attention to the formation of a simple graph where at most one edge between two agents exists.[2] On the other hand, we consider the formation of multiple graphs where multiple edges can exist between two teams, because a team may play with another team

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1 See, for example, Roth and Sotomayor (1990) and Roth (2002), for the survey of the models.

2 Several studies on matching with contracts are exceptions. See, for example, Hatfield and Kominers (2012) and Hatfield et al. (2013).

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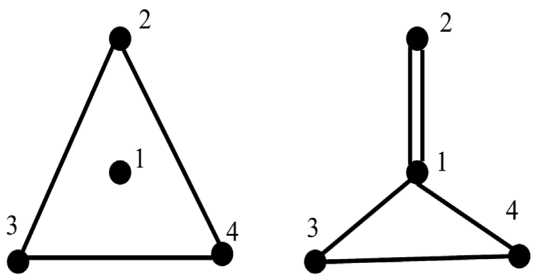

Fig. 1. Preliminary example.

more than once. Moreover, our stability concept is closely related to those of Dutta and Mutuswami (1997) and Jackson and van den Nouweland (2005).

Instead of considering a general matching model, we restrict our attention to a specific preference domain called a *doubly single-peaked domain* . That is, the preference of a team over games played by that team satisfies the following three conditions. First, condition (P1) is that every team has an ideal number of games played with each of the other teams. Second, condition (P2) is that a team is indifferent between any two matchings in which it has the same total number of games and the number of the games played with any other team is not over the corresponding ideal number. Third, condition (P3) is that in the same case, each team has the ideal total number of games and prefers the matching whose total number of games is closest to the ideal total number.

The preference domains that are similar to the doubly single-peaked preference domain are discussed by several works on coalition formation such as Banerjee et al. (2001) and Bogomolnaia and Jackson (2002). Moreover, our model is a generalization of the roommates model with dichotomous preferences introduced by Chung (2000) and one of our results is also a generalization of that of Chung (2000). Bogomolnaia and Moulin (2004) and Roth et al. (2005) also consider the model and the latter applies it to an analysis of a pairwise kidney exchange problem.[3]

In this study, we provide a characterization of individually rational matchings and that of stable matchings in this class. We consider two stability concepts called pairwise stability and group stability. The latter concept is stronger than the former. However, we show that in the doubly single-peaked domain, the necessary and sufficient stability conditions are the same between the two stability concepts.

A matching is said to be constrained efficient if it is not Pareto dominated by any individually rational matching. We show that a stable matching may not be constrained efficient. This implies that a realized matching may not be efficient. Thus, a centralized matching mechanism to obtain a stable and constrained efficient matching is necessary.

We represent a matching as a graph as of that in Fig. 1. Specifically, we focus on a subgraph of a compatibility graph, which reflects the ideal numbers of the games between each pair of teams. Moreover, we consider a graph in which the degree of each node i is less than or equal to a certain positive integer c_i. Such a subgraph is called a c-matching of a compatibility graph where $c = (c_1, c_2, \ldots, c_n) \in \mathbb{Z}^n_+$ and n is the number of teams. Then, we define $c^{**} = \left(c_1^{**}, c_2^{**}, \ldots, c_n^{**}\right)$ where c_i^{**} reflects the preference of team i and a c^{**}-matching of a compatible graph. We show that the set of c^{**}-matchings of a compatible graph is equivalent to that of individually rational matchings. Moreover, a maximum c-matching is defined as a c-matching with the largest number of edges.[4] We show that a maximum c^{**}-matching is stable and constrained efficient. Since a computationally efficient method to derive a maximum c-matching of a graph is known, we can derive a stable and efficient matching within a reasonable time.

Finally, we provide strategic implications of our model. We consider a direct mechanism whereby the teams first reveal their preferences. Unfortunately, we have a negative result. That is, any direct mechanism for selecting a stable and constrained efficient matching does not satisfy strategy-proofness even under the doubly single-peaked preference domain. Hatfield et al. (2014) also have a similar negative result to ours, but they restrict their attention to the max–min preference domain that is independent from ours.

2. Model

Let $N = \{1, 2, \ldots, n\}$ be a set of sports teams. A team can play several games with another team. Let $\mu_{ij} \in \mathbb{Z}^1_+$ be the number of games played between i and j where $\mu_{ij} = \mu_{ji}$ is satisfied for all $i, j \in N$ and $\mu_i = (\mu_{i1}, \mu_{i2}, \ldots, \mu_{in}) \in \mathbb{Z}^n_+$. We assume $\mu_{ii} = 0$ for all $i \in N$. Furthermore, let μ be the $n \times n$ matrix whose (i, j)th entry is μ_{ij}. We call μ a matching. Let $c_i(\mu) = \sum_{j \in N} \mu_{ij}$ that represents the number of games played by i in μ.

We introduce some specific cases of this model. Suppose that N is divided into two disjoint sets F and M where $F \subset N$ and $M = N \diagdown F$, and $\bar{\mu}_{ij} = 0$ must be satisfied for any $i, j \in F$ or $i, j \in M$. Then, this specific case is called the *two-sided case*. Moreover, if $\bar{\mu}_{ij} \leq 1$ for all $i, j \in N$ must hold, then this case is called the *at-most-one-game case*. In the two-sided and at-most-one-game case, this model is equivalent to the two-sided many-to-many model discussed by many previous studies.

We consider preferences of teams. Team i has no concern for μ_{jk} for any $j, k \neq i$ but is concerned with μ_i. That is, we assume a preference relation of team i over $\mathbb{Z}^n_+$ denoted by $\succsim_i$, where $\mu_i \succsim_i (\succ_i)\ \mu_i'$ implies that i weakly (strictly) prefers μ_i to μ_i'. Additionally, $\mu_i \sim_i \mu_i'$ indicates that i is indifferent between μ_i and μ_i'.

We specify that the preference relation of i satisfies a single-peaked preference over the number of games it plays where $c_i^* \in \mathbb{Z}^1_{++}$ represents the ideal total number of games of i. Moreover, we also assume that the preference relation of i is also single-peaked over the number of games that i plays with j. Let $\bar{\mu}_{\overrightarrow{ij}} \in \mathbb{Z}^1_+$ represent i's ideal number of games played with j.[5] Therefore, we call the preference domain focused on in this study a *doubly single-peaked domain*. To be precise, we assume the following conditions.

(P1) If $\mu_{ij} \geq \bar{\mu}_{\overrightarrow{ij}}$, then $\mu_i \succ_i (\mu_{i1}, \ldots, \mu_{ij} + 1, \ldots, \mu_{in})$.
(P2) Suppose that μ and μ' satisfy $\mu_{ij} \leq \bar{\mu}_{\overrightarrow{ij}}$ and $\mu'_{ij} \leq \bar{\mu}_{\overrightarrow{ij}}$ for all $j \neq i$, respectively. If $c_i(\mu) = c_i\left(\mu'\right)$, then $\mu_i \sim_i \mu_i'$.
(P3) Suppose that μ and μ' satisfy $\mu_{ij} \leq \bar{\mu}_{\overrightarrow{ij}}$ and $\mu'_{ij} \leq \bar{\mu}_{\overrightarrow{ij}}$ for all $j \neq i$, respectively. If $c_i\left(\mu'\right) < c_i(\mu) \leq c_i^*$ or $c_i^* \leq c_i(\mu) < c_i\left(\mu'\right)$, then $\mu_i \succ_i \mu_i'$.

The first condition (P1) implies that if $\mu_{ij} \geq \bar{\mu}_{\overrightarrow{ij}}$, then any addition of one game between i and j decreases the utility of team i. This reflects the taste of i for an opponent team represented by j. For example, if team i wants to play with j as many times as possible, then $\bar{\mu}_{\overrightarrow{ij}}$ is a sufficiently large number such as $\bar{\mu}_{\overrightarrow{ij}} = c_i^*$. On the other hand, if i does not want to play even one game with

[3] See also Yilmaz (2011), Okumura (2014), Sönmez and Ünver (2014), Anderson et al. (2015) and Andersson (2015), and Nicolò and Rodríguez-Álvarez (2017) with regard to the applications.

[4] Berge (1985, Ch. 8) defines and discusses the c-matchings and maximum c -matchings.

[5] In this model, $\bar{\mu}_{\overrightarrow{ij}} \neq \bar{\mu}_{\overrightarrow{ji}}$ can be satisfied. For example, consider the case where each team wants to play a game with a strong team. If i is clearly stronger than j, then $\bar{\mu}_{\overrightarrow{ij}} < \bar{\mu}_{\overrightarrow{ji}}$ would be satisfied.