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Fair and square: Cake-cutting in two dimensions

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HIGHLIGHTS

- The cake-cutting problem is extended by adding 2-dimensional geometric constraints.
- In addition to a fair value, the allotted pieces must have a usable geometric shape.
- Several shapes are examined, focusing on squares and balanced aspect-ratio rectangles.
- Our impossibility results show upper bounds on the attainable fair value per agent.
- Our division procedures give each agent a usable plot worth at least half that value.

GRAPHICAL ABSTRACT

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ABSTRACT

We consider the classic problem of fairly dividing a heterogeneous good ("cake") among several agents with different valuations. Classic cake-cutting procedures either allocate each agent a collection of disconnected pieces, or assume that the cake is a one-dimensional interval. In practice, however, the two-dimensional shape of the allotted pieces is important. In particular, when building a house or designing an advertisement in printed or electronic media, squares are more usable than long and narrow rectangles. We thus introduce and study the problem of fair two-dimensional division wherein the allotted pieces must be of some restricted two-dimensional geometric shape(s), particularly squares and fat rectangles. Adding such geometric constraints re-opens most questions and challenges related to cake-cutting. Indeed, even the most elementary fairness criterion – *proportionality* – can no longer be guaranteed. In this paper we thus examine the level of proportionality that *can* be guaranteed, providing both impossibility results and constructive division procedures.

1. Introduction

Fair division of land has been an important issue since the dawn of history. One of the classic fair division procedures, "I cut and you choose", is already alluded to in the Bible (Genesis 13) as a method for dividing land between two people. The modern study of this problem, commonly termed *cake cutting*, began in the 1940's. The first challenge was conceptual — how should "fairness" be defined when the cake is heterogeneous and different people may assign different values to subsets of the cake? Steinhaus (1948) introduced the elementary and most basic fairness requirement, now termed *proportionality*: each of the n agents should get a piece which he values as worth at least $1/n$ of the value of the entire cake. He also presented a procedure, suggested by Banach and Knaster, for proportionally dividing a cake among an arbitrary number of agents. Since then, many other desirable properties

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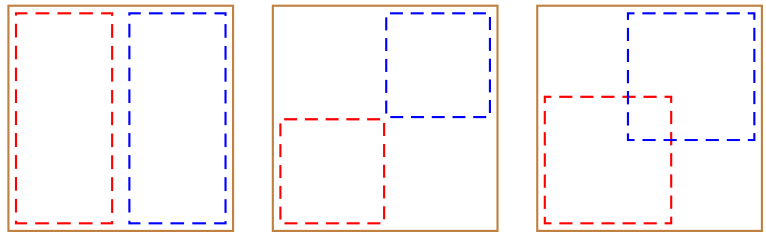

(a) Two disjoint rectangles worth 1/2.

(b) Two disjoint squares worth 1/4.

(c) No two disjoint squares worth more than 1/4.

Fig. 1. Dividing a square cake to two agents.

of cake partitions have been studied, including: envy-freeness (e.g. Weller, 1985; Brams and Taylor, 1996; Su, 1999; Barbanel and Brams, 2004), social welfare maximization (e.g. Cohler et al., 2011; Bei et al., 2012; Caragiannis et al., 2012) and strategy-proofness (e.g. Mossel and Tamuz, 2010; Chen et al., 2013; Cole et al., 2013). See the books by Brams and Taylor (1996), Robertson and Webb (1998), Barbanel (2005), Brams (2007) and a recent survey by Procaccia (2015) for more information.

Many economists regard land division as an important application of division procedures (e.g. Berliant and Raa, 1988; Berliant et al., 1992; Legut et al., 1994; Chambers, 2005; Dall'Aglio and Maccheroni, 2009; Hüsseinov, 2011; Nicolò et al., 2012). Hence, they note the importance of imposing some geometric constraints on the pieces allotted to the agents. The most well-studied constraint is *connectivity* — each agent should receive a single connected piece. The cake is usually assumed to be the one-dimensional interval $[0, 1]$ and the allotted pieces are sub-intervals (e.g. Stromquist, 1980; Su, 1999; Nicolò and Yu, 2008; Azrieli and Shmaya, 2014). This assumption is usually justified by the reasoning that higher-dimensional settings can always be projected onto one dimension, and hence fairness in one dimension implies fairness in higher dimensions.[1] However, projecting back from the one dimension, the resulting two-dimensional plots are thin rectangular slivers, of little use in most practical applications; it is hard to build a house on a 10×1000 m plot even though its area is a full hectare, and a thin 0.1-inch wide advertisement space would ill-serve most advertises regardless of its height.

We claim that the *two-dimensional shape* of the allotted piece is of prime importance. Hence, we seek divisions in which the allotted pieces must be of some restricted family of "usable" two-dimensional shapes, e.g. squares or polygons of balanced length/width ratio.

Adding a two-dimensional geometric constraint re-opens most questions and challenges related to cake-cutting. Indeed, even the elementary proportionality criterion can no longer be guaranteed.

Example 1.1. A homogeneous square land-estate has to be divided between two heirs. Each heir wants to use his share for building a house with as large an area as possible, so the utility of each heir equals the area of the largest house that fits in his piece (see Fig. 1). If the houses can be rectangular, then it is possible to give each heir 1/2 of the total utility (a); if the houses must be square, it is possible to give each heir 1/4 of the total utility (b) but impossible to give both heirs *more* than 1/4 the total utility (c). In particular, when the allotted pieces must be square, a proportional division does not exist.[2]

This example invokes several questions. What happens when the land-estate is heterogeneous and each agent has a different utility function? Is it always possible to give each agent a 2-by-1 rectangle worth for him at least 1/2 the total value? Is it always possible to give each agent a square worth for him at least 1/4 the total value? Is it even possible to guarantee a positive fraction of the total value? If it is possible, what division procedures can be used? How does the answer change when there are more than two agents? Such questions are the topic of the present paper.

We use the term *proportionality* to describe the fraction that can be guaranteed to every agent. So when the shape of the pieces is unrestricted, the proportionality is always $1/n$, but when the shape is restricted, the proportionality might be smaller. Naturally, the attainable proportionality depends on both the shape of the cake and the desired shape of the allotted pieces. For every combination of cake shape and piece shape, one can prove *impossibility results* (for proportionality levels that cannot be guaranteed) and *possibility results*(for the proportionality that can be guaranteed). While we examined many such combinations, the present paper focuses on several representative scenarios which, in our opinion, demonstrate the richness of the two-dimensional cake-cutting task.

1.1. Walls and unbounded cakes

In Example 1.1, the two pieces had to be contained in the square cake. One can think of this situation as dividing a square island surrounded in all directions by sea, or a square land-estate surrounded by 4 walls: no land-plot can overlap the sea or cross a wall.

In practical situations, land-estates often have less than 4 walls. For example, consider a square land-estate that is bounded by sea to the west and north but opens to a desert to the east and south. Allocated land-plots may not flow over the sea shore, but they may flow over the borders to the desert.

Cakes with less than 4 walls can also be considered as unbounded cakes. For example, the above-mentioned land-estate with 2 walls can be considered a quarter-plane. The total value of the cake is assumed to be finite even when the cake is unbounded. When considering unbounded cakes, the pieces are allowed to be "generalized squares" with an infinite side-length. For example, when the cake is a quarter-plane (a square with 2 walls), we allow the pieces to be squares or quarter-planes. When the cake is a half-plane (a square with 1 wall), we also allow the pieces to be half-planes, etc. The terms "square with 2 walls" and "quarter-plane" are used interchangeably throughout the paper.

1.2. Fat objects

Intuitively, a piece of cake is usable if its lengths in all dimensions are balanced — it is not too long in one dimension and too short in another dimension. This intuition is captured by the concept of *fatness*, which we adapt from the computational geometry literature (e.g. Agarwal et al., 1995; Katz, 1997):

[1] In the words of Woodall (1980): "the cake is simply a compact interval which without loss of generality I shall take to be [0, 1]. If you find this thought unappetizing, by all means think of a three-dimensional cake. Each point P of division of my cake will then define a plane of division of your cake: namely, the plane through P orthogonal to [0, 1]".

[2] Berliant and Dunz (2004) use a very similar example to prove the nonexistence of a competitive equilibrium when the pieces must be square.