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Comparing optimal choices with multi-dimensional action spaces

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HIGHLIGHTS

- Lattice methods are used to study a player's multi-dimensional best responses.
- Conditions guarantee that one component will be favored over another at an optimum.
- Helps analyze set of optimal solutions, especially in absence of closed-form solutions.
- Applications to multi-market monopoly and non-differential optimization are given.

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ABSTRACT

This paper introduces ordinal conditions on payoff functions for models with multi-dimensional action spaces which guarantee that the optimal action in one direction is greater than the optimal action in another direction at an optimum. Examples are given to motivate these conditions.

1. Introduction

In many situations, an agent's interaction with her environment consists of multiple interdependent choices. For example, a consumer chooses an optimal bundle by considering various units of individual goods while satisfying a budget constraint. Likewise, a firm operating in two markets chooses how much to charge in one market while considering how this choice will affect profits in the other market. In such cases, it is often of interest to know when an agent will find it optimal to take a higher action in one component relative to another. That is, in the context of the firm, when can it be guaranteed that the firm will choose to charge a higher price in one market relative to the other market? This paper sheds light on the behavioral underpinnings of such situations by establishing order properties directly from an agent's preferences and feasible choices which guarantee that one component will be favored over another.

Our analysis draws on the standard lattice techniques developed in Topkis (1998) which are used in the literature on games of strategic complements (GSC) and substitutes (GSS),[1] and have been extended to consumer theory by Quah (2007). This literature allows for multi-dimensional action spaces to the extent that actions are regarded as single vectors under the product order.[2] This allows one to make statements regarding the monotonicity of the set of optimizers, as is standard in GSC and GSS, for instance. However, lattice methods have not yet been extended to compare the component-wise choices that comprise an optimal choice. That is, while conditions exist under which each component of an optimal choice can be guaranteed to be higher or lower in different environments, the question as to whether one component will be higher or lower than another in a particular environment remains unanswered.

To this end, we introduce ordinal conditions on an agent's preference which are sufficient to guarantee that one component is favored relative to another at an optimal solution. In this sense, our results are broadly related to Lazzati (2013), which derives similar conditions that guarantee when one player will choose a higher strategy than another in a Nash equilibrium. In particular, Lazzati shows that in the context of a GSC, as long as some player i finds it optimal to increase her strategy whenever another player j does, then any fixed point of the extremal best response functions will be such that player i plays a strategy as least as large as player j. Similarly, we show that under some conditions, as long as choosing a higher strategy in some component i is beneficial whenever doing so in another component j is, then the agent will favor component i at an optimizer.

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[1] For a survey article on GSC, see Amir (2005).

[2] Recall that if $(A, \succeq_A)$ and $(B, \succeq_B)$ are two ordered sets, then the product order $\succeq$ on $A \times B$ is defined as, $(a', b') \succeq (a, b)$ if $a' \succeq_A a$ and $b' \succeq_B b$.

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This paper is organized as follows: Section 2 introduces the relevant definitions and theoretical framework. Section 3 contains the results of the paper, while providing several examples. Theorem 1 is the main result, and provides conditions on a payoff function in the case that the constraint set is a lattice which guarantees the agent will favor component over another at an optimizer. Theorem 2 considers the case when the constraint may fail to be a lattice, such as when a consumer is maximizing utility on a budget set.

2. Theoretical framework

We will assume that a decision maker chooses an action from $\mathbb{R}^n$ in order to maximize a payoff function $\pi : \mathbb{R}^n \to \mathbb{R}$. Throughout, we will make use of basic lattice concepts on $\mathbb{R}$. We endow $\mathbb{R}^n$ with the standard product order, so that for $x, y \in \mathbb{R}^n$, $x \geq y$ iff $x_i \geq y_i$, for all $i = 1, 2, \ldots, n$. A subset $S \subset \mathbb{R}^n$ is a **sublattice** if for each $x, y \in \mathbb{R}^n$, $x \vee y$ and $x \wedge y$ are contained in S, where $x \vee y$ and $x \wedge y$ are the supremum and infimum of x and y, respectively, and defined as

$$x \vee y = (\max\{x_1, y_1\}, \max\{x_2, y_2\}, \ldots, \max\{x_n, y_n\})$$
$$x \wedge y = (\min\{x_1, y_1\}, \min\{x_2, y_2\}, \ldots, \min\{x_n, y_n\}).$$

Furthermore, $S \subset \mathbb{R}^n$ is a **quasisublattice** if for each $x, y \in S$, either $x \vee y$ or $x \wedge y$ is contained in S, and a **subcomplete sublattice** if for each non-empty subset $Z \subset S$, the supremum and infimum of Z, denoted $\vee Z$ and $\wedge Z$, respectively, exist and are contained in S. Recall from Topkis (1998) that a sublattice S of $\mathbb{R}^n$ is subcomplete if and only if S is compact.

We will assume that the decision maker chooses from a set of feasible set of actions $\mathscr{F} \subset \mathbb{R}^n$. We will describe a typical element $a \in \mathscr{F}$ as $a = (a_i, a_j, a_{-i,j})$ when we want to emphasize components i and j, where $a_{-i,j}$ represents the choices in all components other than i and j. The decision maker's set of optimal choices is then written as

$$M = \arg\max_{a \in \mathscr{F}} \pi(a).$$

When π is quasisupermodular,[3] then M is a sublattice of $\mathbb{R}^n$ if $\mathscr{F}$ is a sublattice of $\mathbb{R}^n$. Therefore, if π is continuous and $\mathscr{F}$ is compact, them M is a compact sublattice, and hence a subcomplete sublattice, which implies that M contains largest and smallest elements $\vee M$ and $\wedge M$, respectively.

We will also need a way to compare components in the decision maker's feasible action space $\mathscr{F}$. To that end, we will make the following definition:

Definition 1. Let $\mathscr{F} \subset \mathbb{R}^n$. We say that $\mathscr{F}$ **favors component *i* to *j*** if, whenever $a \in \mathscr{F}$ with $a_j > a_i$, then $\tilde{a} \in \mathscr{F}$, where $\tilde{a}_i = a_j$, $\tilde{a}_j = a_i$, and $\tilde{a}_k = a_k$ for all $k \neq i, j$.

Note that $\mathscr{F}$ favoring i to j requires that for all feasible ordered pairs such that the jth component is larger than the ith component, then the "flipped pair" is also feasible. For example, consider a budget set consisting of affordable goods 1 and 2, given by

$$\mathscr{F} = \{(a_1, a_2) \in \mathbb{R}^2_+ \mid p_1 a_1 + p_2 a_2 \leq w\}$$

where w is wealth, and p_1 and p_2 are the prices of goods 1 and 2, respectively. Suppose that $a = (a_1, a_2)$ is affordable, where $a_2 > a_1$. Then as long as $p_2 \geq p_1$, we have that $\tilde{a} = (\tilde{a}_1, \tilde{a}_2)$ is affordable, where $\tilde{a}_1 = a_2$ and $\tilde{a}_2 = a_1$, and hence $\mathscr{F}$ favors 1–2. Note also that $\mathscr{F}$ is a quasisublattice but not a sublattice, since for each $a, a' \in \mathscr{F}$, $a \wedge a' \in \mathscr{F}$, but this is not necessarily true for $a \vee a'$. In the next section, we will explore conditions on π both in the case when $\mathscr{F}$ is a sublattice and then in the case when $\mathscr{F}$ is only a quasisublattice which guarantees that component i is favored to component j at an optimizer.

[3] π is **quasisupermodular** if for each $x, y \in \mathbb{R}^n$, $\pi(y) \geq \pi(x \wedge y) \Rightarrow \pi(x \vee y) \geq \pi(x)$ and $\pi(y) > \pi(x \wedge y) \Rightarrow \pi(x \vee y) > \pi(x)$.

3. Main results

We now explore conditions on the primitives of a model which are sufficient to make a comparison between two components i and j of an optimizer. We will assume throughout this section that $\mathscr{F}$ favors component i to j, according to Definition 1. We first investigate the case when, in addition, $\mathscr{F}$ is a sublattice. We then relax this assumption by allowing for the possibility that $\mathscr{F}$ is a quasisublattice, so that for each $a, a' \in \mathscr{F}$, either $a \wedge a' \in \mathscr{F}$ or $a \vee a' \in \mathscr{F}$, such as with the budget set considered in the previous section. In each case, sufficient conditions which address both the discrete case as well as the differential case are given.

3.1. Sublattice constraint sets

We first consider the case when $\mathscr{F}$ is a sublattice. From now on, we will describe a strategy profile $a \in \mathscr{F}$ as $a = (a_i, a_j, a_{-i,j})$, so that the first entry is the decision maker's choice of action in component i, the second is her choice in component j, and the third her choice in all other components.

We now introduce the following properties, which are similar in nature to Quah and Strulovici's (2009) interval dominance order:

Definition 2. Suppose $\mathscr{F}$ favors component i to j, and let $\pi : \mathscr{F} \to \mathbb{R}$ be a real-valued function. Then

1. π **favors component *i* to *j*** (**of Type 1**) if, for each $a = (z, z', a_{-i,j}) \in \mathscr{F}$ such that $z' > z$, we have that $(z, z, a_{-i,j}) \in \mathscr{F}$, and the following property holds:

   $\pi(z, z', a_{-i,j}) \geq \pi(z, x, a_{-i,j})$
   for all $(z, x, a_{-i,j}) \in \mathscr{F}$ such that x in $[z, z']$
   $\Rightarrow \pi(z', z, a_{-i,j}) \geq \pi(z, z, a_{-i,j})$.[4]

2. π **favors component *i* to *j*** (**of Type 2**) if, for each $a = (z, z', a_{-i,j}) \in \mathscr{F}$ such that $z' > z$, we have that $(z', z', a_{-i,j}) \in \mathscr{F}$, and the following property holds:

   $\pi(z, z', a_{-i,j}) \geq \pi(x, z', a_{-i,j})$
   for all $(x, z', a_{-i,j}) \in \mathscr{F}$ such that x in $[z, z']$
   $\Rightarrow \pi(z', z, a_{-i,j}) \geq \pi(z', z', a_{-i,j})$.

A condition which is sufficient for the Type 1 property can be stated as follows: for each $a = (z, z', a_{-i,j}) \in \mathscr{F}$ such that $z' > z$, we have that $(z, z, a_{-i,j}) \in \mathscr{F}$, and the following property holds:

$$\pi(z, z', a_{-i,j}) \geq \pi(z, z, a_{-i,j}) \Rightarrow \pi(z', z, a_{-i,j}) \geq \pi(z, z, a_{-i,j}). \tag{1}$$

An analogous sufficient condition holds for π favoring component i to j (of Type 2). Definition 2 can be visualized in Fig. 1.

The Type 1 property is satisfied if a positive deviation in the jth direction is beneficial (deviation 1 above), then a similar deviation in the ith direction (deviation 2 above) is beneficial as well. Similarly, the Type 2 property is satisfied if a negative deviation in the ith direction (deviation 1′ above) is beneficial, then a similar deviation (deviation 2′) in the jth direction is beneficial.

The first main result is stated below, which shows that either of the Type 1 or Type 2 properties being satisfied, along with the weak sort of complementarity provided for by quasisupermodularity, is enough to guarantee that an optimizing agent will favor component i over component j.

[4] Note that $(z', z, a_{-i,j}) \in \mathscr{F}$ because $(z, z', a_{-i,j}) \in \mathscr{F}$ and $\mathscr{F}$ favors component i to j.