Journal of Mathematical Economics 71 (2017) 20–27

Contents lists available at ScienceDirect

Journal of Mathematical Economics

journal homepage: www.elsevier.com/locate/jmateco

Utility of wealth with many indivisibilities

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ARTICLE INFO

Article history:
Received 9 September 2016
Received in revised form
31 December 2016
Accepted 28 March 2017
Available online 17 April 2017

Keywords:
Knapsack problem
Friedman–Savage
Expected utility theory
Gambling
Insurance

ABSTRACT

We introduce a class of utility of wealth functions, called knapsack utility functions, which are appropriate for agents who must choose an optimal collection of indivisible goods subject to a spending constraint. We investigate the concavity/convexity and regularity properties of these functions. We find that convexity – and thus a demand for gambling – is the norm, but that the incentive to gamble is more pronounced at low wealth levels. We consider an intertemporal version of the problem in which the agent faces a credit constraint. We find that the agent's utility of wealth function closely resembles a knapsack utility function when the agent's saving rate is low.

1. Introduction

We investigate the properties of the utility of wealth function of an agent who chooses an optimal set of items from among a large number of indivisible items. Each item has a cost and provides some amount of utility to the agent. We are interested in the utility of wealth function obtained by solving this optimization problem for each wealth level. In the computer science literature this optimization problem is called the knapsack problem (Kellerer et al., 2004), so we will refer to the resulting utility function as a *knapsack utility function*.

It has been known since the seminal paper of Friedman and Savage (1948) that an agent who seeks to maximize her expected utility may simultaneously gamble and purchase insurance when her utility function has a region of convexity sandwiched between regions of concavity. There has been a substantial amount of work in finding economic conditions that give rise to utility of wealth functions with convexities (Appelbaum and Katz, 1981; Dobbs, 1988; Hakansson, 1970; Henderson and Hobson, 2013; Jones, 1988; Kwang, 1965; McCaffrey, 1994). Among these, Jones (1988), Kwang (1965), and McCaffrey (1994) have suggested that an indivisibility in the consumption set may induce a region of convexity in the utility of wealth function and from this they recover Friedman and Savage's result that gambling may be part of an optimal utility maximization strategy.

We extend the results of Jones and Kwang to the situation in which all of the consumption goods are indivisible. By considering a model in which there are a large number of indivisibilities, we are able to consider the effect of an agent's wealth on the incentives to gamble that are caused by indivisibilities. In the single-indivisibility models presented by Jones and Kwang, if the agent is wealthy enough that she is past the region of convexity induced by the indivisibility she will prefer not to gamble. However, this conclusion appears to be an artifact of the assumption that there is a single indivisibility. In our model, we find that wealthy agents will tend to see relatively small (but sometimes positive) increases in expected utility from gambling as a result of large scale decreasing marginal utility.

By incorporating a large number of indivisibilities, we find that most wealth levels fall in a region of convexity sandwiched between regions of convexity. As a result, we find that simultaneous gambling and insurance purchase is commonplace. This suggests that trying to predict the agent's behavior with respect to some gamble or contingent liability from a classification of the agent as "risk-loving" or "risk-averse" will likely be unsuccessful. Instead, an understanding of the main items relevant to the agent's consumption decision are necessary for good prediction. While the agent's attitude toward any particular risk depends a great deal on the particulars of the risk, we find that large gambling expenditures and a large monetary value placed on gambling will tend to result from the presence of high-cost, high-utility items that the agent is close to being able to afford.

We consider the applicability of knapsack utility functions to consumer behavior by considering the utility of wealth function in a continuous-time intertemporal model in which the agent's utility of consumption function at each point in time is a knapsack utility function. If the agent is able to borrow freely, the convexities which drive the interesting behavior of knapsack utility functions are

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http://dx.doi.org/10.1016/j.jmateco.2017.03.005
0304-4068/

absent from the utility of wealth function (Jones, 1988). However, after introducing a credit constraint we find that the agent's utility of wealth function converges to a knapsack utility function as the agent's saving rate becomes small. That is, the behavior of a credit-constrained agent with a low saving rate may be closer to that predicted by a knapsack utility function than by a concave utility function. To our knowledge, this is the first paper to use this model to justify transferring convexities in the utility of consumption function to the utility of wealth function.

In the intertemporal model, repeated negative-expected-return gambling may be rational in the sense that each gamble maximizes the agent's expected utility at the time that the agent undertakes it. From a societal perspective, this may be suboptimal because the law of large numbers implies that populations for whom gambling is rational will collectively become poorer. Moreover, members of these populations may find themselves in situations in which repeated participation in negative-expected-value gambles prevents their wealth from increasing with high probability. In short, our results suggest that indivisibility-induced gambling is a kind of poverty trap. In our conclusion, we point out the parameters of our model that could be targeted to reduce incentives for unfavorable gambling.

2. Knapsack utility functions

For us, an instantiation of the knapsack problem is specified by an infinite list of items, each with a cost and a utility. The lists $c = \{c_i\}_{i=1}^{\infty}$ and $u = \{u_i\}_{i=1}^{\infty}$ are the costs and utilities of the items. Given a wealth level w, a solution to the knapsack problem is a sequence $(a_i)_{i=1}^{\infty}$ of 0 s and 1 s that maximizes

$$\sum_{i=1}^{\infty} a_i u_i \quad \text{subject to} \quad \sum_{i=1}^{\infty} a_i c_i \leq w.$$

We may define the *utility of wealth function* by

$$U(w) = \sup_a \left\{ \sum_{i=1}^{\infty} a_i u_i : \sum_{i=1}^{\infty} a_i c_i \leq w \right\}$$

where the sup is taken over all lists $a = \{a_i\}_{i=1}^{\infty}$ with $a_i \in \{0, 1\}$.

We will make various assumptions about the collection of items over the course of the paper. Define the *utility density* of item i to be $d_i = u_i/c_i$.

Assumptions. (i) For all i, $c_i > 0$ and $u_i \geq 0$.
(ii) $d_i \to 0$ and $d_1 > d_2 > \cdots$.
(iii) There is some $C > 0$ for which $c_i \leq C$ for all i.

Assumption (i) states that the agent is a buyer rather than a seller and that all of the goods are positive goods.

Assumption (ii) states that the utility densities tend to 0. The assumption that $d_i \to 0$ implies that we may order our items so that $d_1 \geq d_2 \geq \cdots$. This assumption is the analogue of the typical assumption of decreasing marginal utility of wealth. We insist that this sequence be strictly decreasing to make our results easier to state and prove.

Assumption (iii) asserts that the costs of items are bounded. If we regard indivisibilities as market imperfections that impede trade, we would expect for mechanisms to arise that divide these indivisible items, causing very large indivisibilities to be uncommon. We will not use this assumption for the majority of the paper. The primary importance of assumption (iii) is that it and assumption (ii) together imply that $u_i \to 0$.

Definition 2.1. We will call a function U a *knapsack utility function* if it is the utility of wealth function for a knapsack problem with item set satisfying assumptions (i)–(ii).

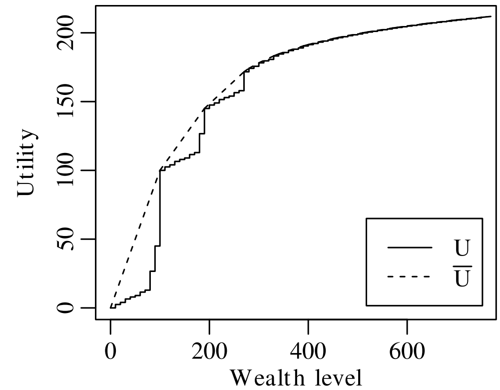

Fig. 1. A knapsack utility function and its convex hull.

A related problem that is useful as a benchmark is the *linear relaxation* of the knapsack problem. In this problem, the constraint that we either buy or not buy a given item is relaxed so that we are allowed to buy any fraction of an item. The utility of wealth function in the linear relaxation of the knapsack problem is given by

$$\bar{U}(w) = \sup_a \left\{ \sum_{i=1}^{\infty} a_i u_i : \sum_{i=1}^{\infty} a_i c_i \leq w \right\},$$

where now the supremum is taken over all lists $a = \{a_i\}_{i=1}^{\infty}$ where $a_i \in [0, 1]$. The greedy algorithm is optimal for this problem. This is the algorithm that puts as much money as possible into the item with the highest utility density, then proceeds to the second highest utility density item and so on. We insist on assumption (ii) in the definition of a knapsack utility function precisely because it guarantees that the greedy algorithm is well-defined.

It follows that $\bar{U}$ is concave. In fact, $\bar{U}$ is the convex hull of the function U. That is, $\bar{U}$ is the smallest concave function larger than U. Since $\bar{U}(w) \leq d_1 w$ for all w, we are able to conclude that $U(w) < \infty$ for all w. That is, our optimization problem never becomes infinite.

In Fig. 1, we show a knapsack utility function with its convex hull. This figure illustrates the features of knapsack utility functions that we find most interesting. First, the frequent oscillation between concavity and convexity can lead to concurrent gambling and insurance purchase as in Friedman and Savage (1948). This is because the agent wants the next large purchase and is fearful of losing her last large purchase. Second, this oscillation is more pronounced at low wealth levels. This is caused by the fact that a wealthy agent will have already moved past the largest jumps in her utility function. That is, the agent will have already taken advantage of highest utility density opportunities.

3. Insurance and gambling

In this section, we will see that the concurrent purchase of insurance and gambles that motivated Friedman–Savage utility functions is not unusual when our agent uses a knapsack utility function. Our first result says that for most wealth levels, there is some contingent liability which the agent is willing to pay a premium to insure against. Our second result says that for most wealth levels, there is some gamble that the agent is willing to pay to accept. Putting the two results together, we find that the potential for concurrent purchase of gambling and insurance exists most of the time.

Definition 3.1. A *contingent liability* is a random variable L such that $L \leq 0$ and $E[L] < 0$. An *insurance policy* against the contingent