Journal of Mathematical Economics 71 (2017) 119–128

Contents lists available at ScienceDirect

Journal of Mathematical Economics

journal homepage: www.elsevier.com/locate/jmateco

Greater parametric downside risk aversion

Donald C. Keenan [a,*,1], Arthur Snow [b]

[a] Department of Economics and Management, Université de Cergy-Pontoise & THEMA,Cergy-Pontoise, 95011, France
[b] Department of Economics, University of Georgia, Athens, GA, 30602, USA

ARTICLE INFO

Article history:
Received 24 December 2016
Received in revised form 12 April 2017
Accepted 10 May 2017
Available online 25 May 2017

Keywords:
Downside risk aversion
Schwarzian
Cautiousness
Protective puts
Self-protection

ABSTRACT

We show that, just as an expected utility maximizer with utility function u responds to a compensated increase in risk by adjusting a control variable to reduce the degree of risk aversion measured by the Arrow–Pratt index $R_u = -u''/u'$ (Diamond & Stiglitz, 1974), so the response to a compensated increase in downside risk entails adjusting the control to reduce the degree of downside risk aversion measured by the Schwarzian $S_u = u'''/u' - (3/2)R_u^2$. We also show that, ceteris paribus, increases in S_u and in R_u result in reduced exposure to downside risk and, therefore, greater demand for self-protection activities that reduce downside risk to future income. An increase from S_u to S_v is characterized by downside risk-averse transformations of utility everywhere along a path from u to v, which together constitute what we define to be a parametric increase in downside risk aversion. These parametric increases yield comparative statics predictions not true if v is simply a downside risk-averse transformation of u, and predictions for incremental changes in risk preferences can be extended immediately to global changes.

1. Introduction

In relating increases in risk and in risk aversion, Diamond and Stiglitz (1974, Theorem 2) establish the fundamental result that the optimal choice for a control variable changes so as to reduce the degree of risk aversion in response to a compensated increase in risk, that is, a change in the distribution for income y that induces a mean preserving spread in the distribution for utility $u(y)$. We extend this result from the second to the third order, showing that the optimal control changes to reduce the degree of downside risk aversion in response to a compensated increase in downside risk, one that preserves the mean and variance of utility.[2]

Diamond and Stiglitz also analyze the case in which the roles of the control and shift variables are reversed, interpreting the control as a preference parameter with the distribution chosen optimally from a family of risks. However, when the family is ordered by degree of risk, all risk averters choose the least risky distribution. By contrast, when ordered by degree of downside risk, an interior distribution may be optimal for some decision makers. We show that, in these instances, an increase in downside risk aversion and in risk aversion leads to a reduction in exposure to downside risk. In the context of self-protection activities that reduce exposure to future downside risk, the implication is that demand for these activities increases with increases in downside risk aversion and in risk aversion.

These conclusions are predicated on the degree of risk aversion for utility $u(y)$ being measured by the index of absolute risk aversion $R_u = -u''/u'$ introduced by Arrow (1970) and Pratt (1964), and on the degree of downside risk aversion being measured by the Schwarzian index $S_u = u'''/u' - (3/2)R_u^2$, which we introduced in Keenan and Snow (2002). These two measures share an important mathematical property responsible for their centrality in the comparative statics predictions summarized above, namely, both satisfy 1-cocycle conditions that convert a composition mapping, such as $v = \varphi(u)$, into the addition operation through

$$R_v - R_u = u'R_\varphi \tag{1}$$

in the case of risk aversion, and through

$$S_v - S_u = [u']^2 S_\varphi \tag{2}$$

in the case of downside risk aversion. These conditions ensure that rankings based on R_φ and on S_φ are transitive and are therefore capable of producing meaningful comparative statics predictions.[3]

* Corresponding author.
E-mail addresses: dkeenan@uga.edu (D.C. Keenan), snow@uga.edu (A. Snow).
1 DCK acknowledges the support of LABEX MME D-II.
2 Rothschild and Stiglitz (1970) define an increase in risk to be a mean preserving spread of the income risk and show these are disliked by all risk averters, for whom $u'' < 0$. (Note that we use primes throughout to denote derivatives.) At the third order, Menezes et al. (1980) define an increase in downside risk to be a mean-and-variance preserving spread of the income risk and show that these are disliked by all downside risk averters, for whom $u''' > 0$.

3 After replacing u with φ_1, φ with φ_2, and v with $\varphi_3 = \varphi_1 \circ \varphi_2$, conditions (1) and (2) show that rankings by R_φ and by S_φ are preserved under composition of

http://dx.doi.org/10.1016/j.jmateco.2017.05.002
0304-4068/

As shown in Theorem 1 of Keenan and Snow (2002), an infinitesimal change in risk preferences uniformly increases the Schwarzian if and only if the transformation of utilities is itself downside risk averse, just as the index of risk aversion increases uniformly if and only if the transformation is risk averse. Indeed, it seems natural to associate greater downside risk aversion with downside risk-averse transformations of utility as an intuitive extension of the association of greater risk aversion with risk-averse transformations. However, the equivalence of increases in the Schwarzian and downside risk-averse transformations (ones with $\varphi''' > 0$) does not extend to large changes in risk preferences. In the large, such a transformation of utility does not continue to imply an increased Schwarzian measure, since unlike the ordering induced by the Schwarzian, the ranking of utility functions induced by downside risk-averse transformations is not even transitive, and so cannot serve as the basis for a coherent definition of greater downside risk aversion.

In this paper, we equate greater downside risk aversion with an increasing Schwarzian by requiring that utility $v(y)$ differ from $u(y)$ by a family of transformations each of which is downside risk averse, even at the infinitesimal level. We refer to this as a parametric increase in downside risk aversion. It is straightforward to show that $S_v > S_u$, or equivalently $S_\varphi > 0$, necessitates that v is a downside risk-averse transformation of u, but the converse, that $\varphi''' > 0$ implies $S_v > S_u$, is not true in the large, since the ranking by $\varphi''' > 0$ is not necessarily transitive but the ranking by S_φ is. We show, that $S_v > S_u$ is, instead, equivalent to a parametric increase in downside risk aversion. The restriction to parametric increases constrains the ranking by φ''' in precisely the manner needed to produce an ordering, by extending to the large the consistency of the latter in the small. For these parametric increases in downside risk aversion, where S_u is increasing everywhere along a path from u to v, comparative statics predictions obtained for small changes in risk preferences extend immediately to predictions for global changes, a property we exploit in establishing the predictions described above.

Parametric increases in downside risk aversion are formally introduced in the next section. In Section 3, we apply these results by extending the comparative statics predictions developed by Diamond and Stiglitz to increases in downside risk and in downside risk aversion. In Section 4, we analyze a thought experiment that cannot be examined at the second order concerning the choice of exposure to risk. Relationships with the literature are discussed in Section 5 and conclusions are offered in Section 6.

2. Parametric increases in downside risk aversion

As suggested in the introduction, it is important to recognize that the condition $S_\varphi > 0$ differs from $\varphi''' > 0$ only for finite changes in risk preferences. For infinitesimal changes, the two conditions are one and the same. To investigate infinitesimal changes in risk preferences we first posit a parameterized family of transformations $\varphi(u(y), c)$ of utility $u(y)$, giving rise to a family of utility functions $u(y, c) \equiv \varphi(u(y), c)$. An infinitesimal increase in c is represented by the transformation

$$\tilde{\varphi}(u(y, c), \tilde{c}) = u(y, c + \tilde{c}). \tag{3}$$

so that $\partial\tilde{\varphi}/\partial\tilde{c}|_{\tilde{c}=0}$ gives the effect on $u(y, c)$ of an infinitesimal increase in c.

Define $S_\varphi(u(y), c)$ to be the Schwarzian for $\varphi(u(y), c)$. The effects of an infinitesimal change in risk preferences on S_φ, and on φ and its utility derivatives, are given by their partial derivatives with respect to c.

mappings, and are therefore transitive, implying that the rankings are strict partial orderings.

Lemma 1. *An infinitesimal change in risk preferences has exactly the same effect on S_φ as on φ'''.*

Proof. Observe that, as a consequence of (3), we have $\tilde{\varphi}'(u(y, c), 0) = 1$, $\tilde{\varphi}''(u(y, c), 0) = 0$, and $\tilde{\varphi}'''(u(y, c), 0) = 0$. Using these equations, straightforward differentiation of $S_{\tilde{\varphi}} = \tilde{\varphi}'''/\tilde{\varphi}' - 3/2(\tilde{\varphi}''/\tilde{\varphi}')^2$ with respect to $\tilde{c}$ yields

$$\frac{\partial S_{\tilde{\varphi}}(u(y, c), \tilde{c})}{\partial \tilde{c}}\Big|_{\tilde{c}=0} = \frac{\partial \tilde{\varphi}'''(u(y, c), \tilde{c})}{\partial \tilde{c}}\Big|_{\tilde{c}=0}, \tag{4}$$

indicating that the effect of an infinitesimal change in preferences is the same for $S_{\tilde{\varphi}}$ and $\tilde{\varphi}'''$. Finally, observe that $\tilde{\varphi}(u(y, c), 0) = u(y, c) = \varphi(u(y, c))$, which implies $S_\varphi(u(y, c), 0) = S_\varphi(u(y), c)$. Hence, $\frac{\partial S_{\tilde{\varphi}}(u(y,c),\tilde{c})}{\partial \tilde{c}}\big|_{\tilde{c}=0} = \frac{\partial S_\varphi(u(y),c)}{\partial c}$ and $\frac{\partial \tilde{\varphi}'''(u(y,c),\tilde{c})}{\partial \tilde{c}}\big|_{\tilde{c}=0} = \frac{\partial \varphi'''(u(y),c)}{\partial c}$, so that the effect is the same for S_φ and φ'''. □

Since $S_{\tilde{\varphi}}(u(y, c), 0) = 0 = \tilde{\varphi}'''(u(y, c), 0)$, Eq. (4) implies that an infinitesimal transformation of risk preferences is downside risk averse if and only if the Schwarzian of that transformation is positive. Therefore, in the small, the analogy between $S_\varphi > 0$ and $R_\varphi > 0$ is exact, as the latter is equivalent to $\varphi'' < 0$ and the former is equivalent to $\varphi''' > 0$. That is, $R_\varphi > 0$ increases risk aversion and $S_\varphi > 0$ increases downside risk aversion in the manner that accords with intuition. However, in the large, while risk-averse transformations remain equivalent to global increases in R_u, a downside risk-averse transformation continues to be implied by but no longer implies an increase in S_u. This asymmetry disappears, though, if, as is quite usual in an economic context, one looks at parametric families of changes in risk preferences. As shown in the Appendix, an increase in risk aversion between u and v, in the sense of $R_\varphi > 0$, is equivalent to R_u increasing along a smooth path of utility functions between the two. As we shall see, exactly the same is true with regard to $S_\varphi > 0$ and increases in the downside risk aversion measure S_u along a path.

Definition 1. Utility $v(y)$ is *parametrically more downside risk averse* than $u(y)$ *along a given smooth path* (of utility functions) if given a parameterization of that path $u(y, c)$ for $c \in [0, \bar{c}]$, so that $u(y) = u(y, 0)$ and $v(y) = u(y, \bar{c})$, we have $\partial\tilde{\varphi}'''(u(y, c), \tilde{c})/\partial\tilde{c}|_{\tilde{c}=0} > 0$ for all $c \in [0, \bar{c})$, with $\tilde{\varphi}$ as given by (3).

This definition requires a positive value for the right-hand side of Eq. (4) everywhere along the path, indicating, by (2), that S_u increases everywhere along the path. Defining $S_u(y, c)$ to be the Schwarzian for $u(y, c)$, we can write

$$\frac{\partial S_u(y, c)}{\partial c} > 0 \tag{5}$$

as the condition for a parametric increase in downside risk aversion. We can now state the following theorem, where the proof of this and other results not in the body are found in Appendix A.

Theorem 1. *Utility $v(y)$ is parametrically more downside risk averse than $u(y)$ along a given smooth path if and only if S is uniformly increasing along the path from u to v. Furthermore, $v(y)$ is parametrically more downside risk averse than $u(y)$ along the path if and only if any function that transforms one utility into another located later along the path is downside risk averse, be the change finite or infinitesimal.*

By controlling infinitesimal changes in utility along a path from u to v so that each utility function along the path is a downside risk-averse transformation of its immediate predecessors, parametric increases in downside risk aversion yield rankings of utility functions by the magnitude of S which, given the 1-cocycle condition (2), provide orderings of utility. In order to speak of an ordering of utility functions alone, we will simply refer to utility $v(y)$ as