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# New characterizations of increasing risk

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ABSTRACT

I present alternative constructions of gambles with greater risk. Rothschild and Stiglitz (1970) demonstrate that gamble $Y$ has greater risk than $X$ when $Y$ is equal in distribution to $X + Z$, where $Z$ is noise. Gambles called *positive-upper-conditional-mean errors* are introduced, and I show that $Y$ has greater risk than $X$ when $Z$ is a PUCME and is not noise. Simple examples demonstrate that the set of PUCMEs is strictly greater than the set of gambles that are noise.



## 0. Introduction

In their fundamental work Rothschild and Stiglitz (RS, 1970) characterize what it means to say that a gamble (random wealth level) $Y$ *has greater risk than* $X$. I extend their work, providing new and alternative characterizations of increasing risk.

RS give a cycle of proofs to demonstrate the equivalence of three statements about the gambles $X$ and $Y$, and their cumulative distribution functions $F$ and $G$. Let $u$ be an investor's utility function, and write $Y \underset{d}{=} X + Z$ to say that $Y$ is equal in distribution to $X + Z$. The equivalent statements are: (i) $Y$ *differs from* $X$ *by noise*, which means that a gamble $Z$ exists such that $Y \underset{d}{=} X + Z$ and that satisfies

$$E(Z|X) = 0; \quad (1)$$

(ii) *every risk averter prefers* $X$ *to* $Y$, which formally requires

$$E(u(X)) \geq E(u(Y)) \quad (2)$$

for all investors with concave functions $u$, and I write $X \gtrsim Y$ when this condition is satisfied; and (iii) $G$ *has more weight in the tails than* $F$, which requires that

$$\int_0^1 S(v)\,dv = 0 \quad (3)$$

and

$$\int_0^x S(v)\,dv \geq 0, \quad x \in [0, 1] \quad (4)$$

are satisfied, where $S = G - F$ is the difference in the distribution functions.[1]

Each of these alternatives characterizes increasing risk among gambles. Because RS do not require a parametric family of distributions, their work has been recognized and applied in many areas of research, including: investment choice (Hartman, 1972; Gollier, 2011), search theory (Rothschild, 1974; Nishimura and Ozaki, 2004), the theory of capital structure (Merton, 1974), the economics of capital requirements (Blum, 1999), and decision theory (Baker, 2006).

This paper reconsiders condition (i) regarding noise. It is a fact, recognized by RS, that there exist gambles $Z$ that satisfy $Y \underset{d}{=} X + Z$ and $X \gtrsim Y$, but that do not satisfy (1). In other words, there is a set of gambles that includes but is strictly larger than noise such that $Y$ has greater risk than $X$. I characterize a subset of this larger set of gambles in this paper.

A *positive-upper-conditional-mean error* (*PUCME*) is defined by

$$E(Z) = 0, \quad (5)$$

and

$$E(Z|X \geq x) \geq 0, \quad x \in [0, 1]. \quad (6)$$

If $Y \underset{d}{=} X + Z$ and $Z$ satisfies (5), the means of $X$ and $Y$ are equal, so it is sensible to say that $Z$ is an error. If $Z$ also satisfies (6) then its upper conditional mean is non-negative, and the addition of $Z$ to $X$

[1] In the formal analysis, $X$, $Y$ and $Z$ have ranges $[0, 1]$, $[0, 1]$ and $[-1, 1]$, respectively, and $u$ is bounded and concave on $[0, 1]$. In the example in Section 2, the common range of the gambles is the real line.

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shifts weight to the tails of the distribution so that all risk averters prefer $X$ to $Y$. Values of $Z$ for large $X$ tend to be positive, while those for small $X$ tend to be negative when (5) and (6) are satisfied. Condition (1) implies condition (6), so a gamble that is noise is also a PUCME.[2] However, there are PUCMEs that are not noise.

If $Y \underset{d}{=} X + Z$ and $Z$ is a PUCME, I say $Y$ *differs from* $X$ *by a PUCME*. The following theorem is the principal result of this paper.

**Theorem 1** (*Sufficiency*). *If gamble Y differs from X by a PUCME then* $X \gtrsim Y$. *(Necessity) If* $X \gtrsim Y$ *then Y differs from X by a PUCME.*

RS demonstrate that if $(a)$ *Y differs from X by a PUCME* and $(b)$ *Z is noise*, $X \gtrsim Y$. The statement of Sufficiency says that the condition $(b)$ can be eliminated. A proof of Necessity is trivial. RS show that if $X \gtrsim Y$ then $Y$ differs from $X$ by noise. Because any gamble that is noise is also a PUCME, the result follows.

These facts raise questions regarding uniqueness. Is it always true that gambles $Z$ exist that are PUCMEs but that are *not* noise? That is, in what settings do gambles $X$ and $Y$ exist such that the *only* PUCME $Z$ for which $Y \underset{d}{=} X + Z$ is noise? A partial answer to these questions is immediately clear when $X$ is constant. If $\Pr(X = x) = 1$ then $E(Z) = 0$ only if $E(Z|X = x) = 0$ and $Z$ is noise; there is no other PUCME that satisfies (5). However, two examples demonstrate that noise is not the unique PUCME when $X$ is random.

In the first example, $X$ and $Y$ are normally distributed with equal means and unequal variances, and $X \gtrsim Y$ as a result. A parameterized family of gambles $Z$ is chosen such that $Y \underset{d}{=} X + Z$ and the conditions (5) and (6) are satisfied by each $Z$. One of these gambles satisfies (1), but with measure one in the parameter space, condition (1) does not hold, demonstrating a multiplicity of non-noise PUCMEs. In a second example, $X$ and $Y$ are discrete gambles such that $X \gtrsim Y$. As in the first example, there is a family of gambles $Z$ such that $Y \underset{d}{=} X + Z$, and only one member of the family is noise. This example leads to the conclusion that, within the universe of discrete gambles, noise is the unique PUCME only when $X$ is constant.

In many economic studies, including many of those cited above, individuals prefer more to less wealth. I revise the theory to consider this case, following Hadar and Russell (1969), who say that $X$ stochastically dominates $Y$ (in second order) if (2) holds for all increasing and concave $u$. In the revision, I show that if $Y \underset{d}{=} X + Z$ and $Z$ satisfies both (6) and

$$E(Z) \leq 0, \tag{7}$$

then $X$ stochastically dominates $Y$. Huang and Litzenberger (1988) demonstrate that

$$E(Z|X) \leq 0 \tag{8}$$

is a sufficient condition for stochastic dominance. However, there are many $Z$ that satisfy (6) and (7) that do not satisfy (8).

In sum, the full character of the gamble $Z$ is indeterminate when $X + Z$ is equal in distribution to $Y$ and either: $Y$ has greater risk than $X$; or $X$ stochastically dominates $Y$. In empirical work, if we observe that all risk averters prefer gamble $X$ to $X + Z$, we should not conclude from that observation alone that $Z$ is noise. Similarly, this paper demonstrates a variety of ways to construct in theory and experimental work a gamble $Y$ that stochastically dominates gamble $X$.

[2] Using (1) and the law of iterated expectations, $E(Z|X \leq x) = E(E(Z|X)|X \leq x) = 0$ when $Z$ is noise.

Section 1 presents the proof of the Sufficiency half of Theorem 1, as well as a corollary that describes the case of stochastic dominance. Sections 2 and 3 present the examples. Section 4 offers final remarks, including a brief example of the theorem applied in a behavioral experiment. Appendix describes an extension of the example in Section 3.

## 1. Adding a PUCME increases risk

Here the gambles $X$, $Y$ and $Z$ may follow discrete, continuous or mixed distributions, while the utility function $u$ is bounded and concave on $[0, 1]$. Let $M(X) = E(Z|X)$. Using the law of iterated expectations, the conditions (5) and (6) that define a PUCME are equivalent to

$$E(M(X)) = 0, \tag{9}$$

and

$$E(M(X)|X \geq x) \geq 0, \quad x \in [0, 1], \tag{10}$$

respectively. The Sufficiency part of Theorem 1 is alternatively stated as

**Theorem 1*** (*Sufficiency*). *If* $Y \underset{d}{=} X + Z$ *and gamble Z satisfies conditions (9) and (10), where* $M(X) = E(Z|X)$, *then* $X \gtrsim Y$.

**Proof of Theorem 1*.** We must demonstrate $E(u(X) - u(X+Z)) \geq 0$. To do so, let $\{a_{n,i}\}$, $i = 0, \ldots, n$, be a sequence of increasingly finer partitions of $[0, 1]$ indexed by $n$. For each $n$, $n = 1, 2, \ldots$, let $a_{n,0} = 0$, $a_{n,n} = 1$, and $a_{n,i} < a_{n,i+1}$, $i = 0, \ldots, n-1$, and let the norms of the partitions converge to zero, which means $\lim_{n\to\infty} \max_i (a_{n,i+1} - a_{n,i}) = 0$. One such sequence of partitions, for example, has elements $a_{n,0} = 0$ and $a_{n,i} = 2^{i-n}$ otherwise for all $n$.

For each $n$, let $u_n$ be the continuous and piece-wise linear function constructed from the secants of $u$ with knots defined by the elements of the partition $\{a_{n,i}\}$. For each $n$, this function is

$$u_n(x) = u(a_{n,i}) + \kappa_n(x)(x - a_{n,i}), \quad x \in [a_{n,i}, a_{n,i+1}),$$

where

$$\kappa_n(x) = \frac{u(a_{n,i+1}) - u(a_{n,i})}{a_{n,i+1} - a_{n,i}}, \quad x \in [a_{n,i}, a_{n,i+1}),$$

for each $i = 0, \ldots, n-1$, and we set $u_n(1) = u(1)$ and $\kappa_n(1) = \kappa_n(a_{n,n-1})$. Because $u$ is concave, the slopes of the secants satisfy

$$\kappa_n(x) \geq \kappa_n(y), \quad x \leq y,$$

and each $u_n$ is a concave function as a result. For this reason,

$$u_n(x+z) \leq u_n(x) + \kappa_n(x) z, \tag{11}$$

for all $x$ and $x + z$ in the interval $[0, 1]$. Furthermore, the $u_n$ converges uniformly to $u$ from below. For each $n$, $u_n(x) \leq u(x)$, $x \in [0, 1]$, and for each $\epsilon > 0$ a $n_\epsilon > 0$ exists such that

$$\sup_{x\in[0,1]} u(x) - u_n(x) < \epsilon, \quad n > n_\epsilon.$$

As a consequence,

$$E(u(X) - u(X+Z)) = \lim_{n\to\infty} E(u_n(X) - u_n(X+Z)).$$

To complete the proof, we demonstrate that the limit on the right is non-negative. We have

$$\begin{aligned} &E(u_n(X) - u_n(X+Z)) \\ &= E(u_n(X) + \kappa_n(X) Z - u_n(X+Z)) - E(\kappa_n(X) Z) \\ &\geq -E(\kappa_n(X) M(X)) \end{aligned}$$