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Optimal mechanisms with simple menus

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ABSTRACT

We consider revenue-optimal mechanism design for the case with one buyer and two items, when the buyer's valuations are independent and additive. We obtain two sets of structural results of the optimal mechanisms, which can be summarized in one conclusion: under certain distributional conditions, the optimal mechanisms have simple menus.

The first set of results states that, under a condition that requires that the types are concentrated on lower values, the optimal menu can be sorted in ascending order. Applying the theorem, we derive a revenue-monotonicity theorem which states that stochastically dominated distributions yield less revenue.

The second set of results states that, under certain conditions which require that types are distributed more evenly or are concentrated on higher values, the optimal mechanisms have a few menu items. Our first result states that, for certain such distributions, the optimal menu contains at most 4 menu items. The condition admits power density functions. Our second result works for a weaker condition, under which the optimal menu contains at most 6 menu items. Our last result in this set works for the unit-demand setting, it states for uniform distributions, the optimal menu contains at most 5 items.

1. Introduction

Revenue-optimal mechanism design has been a topic of intensive research over the past thirty years. The problem is, for a seller, to design a revenue-maximizing mechanism for selling k items to n buyers, given the buyers' distributions of valuations but not the actual values themselves. A special case of the problem, where there is only one item ($k = 1$) and buyers have independent valuation distributions, has been resolved by Myerson's seminal work (Myerson, 1981). Myerson's approach has turned out to be very general and has been applied to a number of similar settings, such as Maskin et al. (1989), Jehiel et al. (1996), Levin (1997), Ledyard (2007) and Deng and Pekeč (2011).

However, this line of work is limited because it does not deepen the understanding of the cases with more than one items ($k > 1$). In fact, even for the simplest multi-item case, where there are two independent items ($k = 2$) and one buyer ($n = 1$) with additive valuations, a direct characterization of the optimal mechanism is still open for general continuous valuation distributions.

For the special case of selling multiple, independent items to a single buyer, significant progress has been made in this particular setting lately. Hart and Nisan (2012) investigate the two simplest forms of auctions: selling the two items separately and selling them as a bundle. They prove that selling separately obtains at least one half of the optimal revenue while bundling always returns at least one half of the separate sale revenue. Hart and Nisan (2013) investigate how the "menu size" of an auction can affect the revenue and show that the revenue of any finite menu-sized auction can be arbitrarily far from the optimal (this implies that restricting attention to deterministic auctions, which have an finite-sized menu, indeed compromises generality). Carrol (2015) considers a robust version of the optimal mechanism design problem, where there is one buyer and multiple additive items and the seller only knows the marginal valuation distributions of each item but not the joint distribution. He shows that the worst-case (with respect to any joint distribution that is consistent with the marginal distributions) optimal mechanism is to separately sell each item via a Myerson auction.

With respect to the literature of exactly optimal mechanism design, Thanassoulis (2004) provides examples where the optimal

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mechanism requires randomized allocations. Pycia (2006) further shows that in general, the optimal mechanism is randomized. Manelli and Vincent (2006, 2007) and Pavlov (2011a,b) obtain optimal mechanisms for several specific distributions (such as when both items are distributed according to the uniform [0, 1] distribution). Daskalakis et al. (2013) and Daskalakis et al. (2015) study this problem from the perspective of duality theory. First they formulate the problem as a maximization problem over a convex domain and then consider its dual in the form of an optimal transportation problem. Their main result is a strong duality theorem, by applying the duality theorem, they can certify optimality by providing a complementary solution to the dual problem. Examples that illustrate their techniques include the optimality of an infinite-menu mechanism for two independent beta distributions, as well as optimality conditions for the grand bundling. Menicucci et al. (2015) prove sufficient conditions under which bundling is optimal for one buyer and two additive items. We will discuss the connection of this paper to our results in Section 5. Haghpanah and Hartline (2015) identify the sufficient and necessary conditions (include the independent uniform case) under which for one unit-demand buyer with two items, the optimal mechanism is to post a price for each item. We will discuss the connection of this paper to our result in the unit-demand section.

In the present paper, we study the case with one buyer and two independent items, in hope of a direct characterization of optimal mechanisms. We obtain several interesting structural results. *Our conclusion is that, under some distributional conditions, optimal mechanisms have "simple" menus.* We summarize our results into two parts, based on the conditions under which the results hold, as well as the different interpretations of "simplicity".

For ease of presentation, we will use the following notation: for a density function h, the *power rate* of h is $PR(h(x)) = \frac{xh'(x)}{h(x)}$.

- Part I (Section 4). If the density functions f_1 and f_2 satisfy $PR(f_1(x)) + PR(f_2(y)) \leq -3, \ \forall x, y$, a condition that roughly states that the types are concentrated on lower values, the optimal mechanism has a monotone menu – sort the menu items in ascending order of payments, the allocation probabilities of both items increase simultaneously – a desirable property that fails to hold in general (cf. Hart and Reny, 2012). Our result complements Hart and Reny's observation and has two important implications.
  1. Hart and Nisan (2012, Theorem 28). Hart and Nisan show that, if the two items are further identically distributed (i.e., $f_1 = f_2$), the bundling auction is optimal. Our result subsumes this theorem as a corollary.
  2. A revenue monotonicity theorem: Based on the menu monotonicity theorem, we are able to prove that, stochastically dominated distributions yield less revenue, another desirable property that fails to hold in general.

  Our proof is constructive and *geometrical* in the sense that we fix the buyer utility function on certain boundary lines of the valuation domain (according to the revenue formula, the seller's revenue is not increasing in the buyer's utility on these boundary lines, thus hard to analyze, so we fix this part of the utility function) and construct the remainder of the optimal utility function (for the remainder part of the valuation domain, the revenue is increasing in the buyer's utility, according to the revenue formula). For several recent applications of the geometrical approach, see Wang and Tang (2015), Tang and Wang (2016) and Tang et al. (2016).
- Part II. (Section 5). If the density functions f_1 and f_2 satisfy $PR(f_1(x)) + PR(f_2(y)) \geq -3 \ \forall x, y$, a condition which roughly asserts that the types are distributed more evenly than the case described in Part I or are concentrated on higher values, the optimal mechanisms contain few menu items. In particular,
  1. If both $PR(f_1(x))$ and $PR(f_2(y))$ are constants, the optimal mechanism contains at most 4 menu items. The result is tight. The constant power rate is satisfied by power functions $h(x) = ax^b$ and the uniform distribution as a special case. This is consistent with earlier results for uniform distributions (Manelli and Vincent, 2006; Pavlov, 2011a): the optimal mechanisms indeed contain four menu items.
  2. – If $PR(f_1(x)) + PR(f_2(y)) = -3 \ \forall x, y$, the optimal mechanism contains at most 3 menu items.
     – If $-2 \leq PR(f_1(x)) \leq y_A f_2(y_A) - 2$ and $-2 \leq PR(f_2(y)) \leq x_A f_1(x_A) - 2$, the optimal mechanism contains 3 menu items. Here x_A and y_A are the lowest possible valuations for items 1 and 2 respectively, Consequently, under either condition, selling the two items as a bundle yields at least half of the optimal revenue.
  3. If we relax the condition to the case where $PR(f(x))$ is monotonically increasing, a fairly general condition satisfied by many distributions, the optimal mechanism is still simple in that it contains at most 6 menu items. This condition includes density functions such as exponential density and any density function whose Taylor series coefficients are nonnegative.
  4. Our last result requires that the buyer demands at most one item. Under this condition, for uniform densities, the optimal mechanism contains at most 5 menu items. The result is also tight.

  These results are in sharp contrast to Hart and Nisan's recent result that there is some distribution where a finite number of menu items cannot guarantee any fraction of revenue (Hart and Nisan, 2013). Here we show that, for several wide classes of distributions, the optimal mechanisms have finite and simple menus. The conditions in these results are necessary; when the conditions do not hold, Daskalakis et al. (2013, 2015) show that, for a setting with two beta distributions, the optimal menu must consist of a continuum of menu items.

  Our proofs for this part are based on Pavlov's characterization and geometrical analysis of how the revenue changes as a function of the utility of the buyer. The intuition is as follows: the "extreme points" in the set of convex utility functions on the boundary values are piecewise linear functions. By standard geometrical arguments, one can further show that these piecewise linear functions only contain a small number of pieces. Since the utility on inner values is linearly related to that on the boundary (because the gradient of the utility function on one direction must be 1 according to Pavlov, 2011a,b), it must be the case that the utility function on the inner points contains only a few linear pieces as well. In other words, the mechanism only contains a few menu items.

In the optimal auction design problem, bidders are utility maximizers. By incentive compatibility, the equilibrium utility as a function of the valuation must be convex. The hardness of optimal auction design is to maximize the seller's revenue under the convex constraints. As one can expect, a common approach is to relax the convex constraint and compute the optimal solution of the relaxed problem. If one is lucky in that the relaxed optimal solution happens to be convex, an optimal solution is found. However this method fails sometimes. As mentioned, Daskalakis et al. (2015) transform the optimal mechanism design to the optimal transportation problem and give a procedure to certify the optimality of the auction. However, difficulties still exist when constructing the optimal solution to the transportation problem.

In parallel with this approach, we adjust the utility function while maintaining the convex constraints. We start from any convex utility, then try to increase or decrease the utility on every point and maintain the convex property in each small region.