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Crisp monetary acts in multiple-priors models of decision under ambiguity

Eric André

emlyon business school, 23 Avenue Guy de Collongue, 69134 Ecully, France

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ABSTRACT

In axiomatic models of decision under ambiguity using a set of priors, a clear distinction can be made between acts which are affected by ambiguity and those which are not: the crisp acts. In these multiple-priors models, the decision maker is indifferent between holding a constant act or holding a non constant crisp act with the same expected utility, if it exists. In financial settings, we show that this indifference, together with the standard definition of monetary acts in the Anscombe–Aumann framework, implies that the investor ignores the variance of some assets, a behavior which conflicts with the assumption on which modern portfolio theory has been built. In this paper we establish the geometrical and topological properties of the set of priors that rule out the existence of non constant crisp acts. These properties in turn restrict what can possibly be an unambiguous financial asset.

1. Introduction

The success of decision theory under ambiguity as a field of research builds upon its ability to rationalize not only individual behaviors that are not well described by the Expected Utility (EU) paradigm, but also empirical features in economics that were at odds with prevailing theories. This is especially true in finance where ambiguity aversion has emerged as one of the possible explanation for some of the well known puzzles of the asset pricing and portfolio choice theories.[1] As a consequence, there is a specific interest for financial applications of axiomatic models of decision under ambiguity, not least as they can help showcase the tractability and idiosyncrasies of these theoretical models.

There are, indeed, a large number of models in the literature, built on different behavioral assumptions and setups. Nonetheless, Cerreia-Vioglio et al. (2011a, henceforth abbreviated CGMMS11) have shown that, for most of them, it is possible to "identify probabilities that are significant for the decision maker's choices regardless of the representation of her preferences". This result highlights the central position of the family of models that generalize the unique probability of the EU with a set of *priors*, family which has developed from the seminal Maxmin Expected Utility (MEU) model of Gilboa and Schmeidler (1989), the subsequent behavioral interpretation of the set of priors being due to Ghirardato et al. (2004).

Formally, both these papers and CGMMS11 have been developed in the framework due to Anscombe and Aumann (1963, henceforth abbreviated AA). To apply these models it is therefore necessary to describe the relevant objects of choice – random variables for financial applications – in this AA framework. Acts are defined as functions from a state space S to a set X of consequences: initially a set of objective lotteries (probability distributions with finite support) and, in subsequent generalizations, a convex subset of a vector space. While the tractability of the AA framework owes a lot to the structure that endows the set X (the mixture operation on lotteries or the algebraic operations of a vector space), the utility function $u: X \to \mathbb{R}$ that is derived in representation theorems is affine. It is then not enough to define the set of consequences X to be the real line to model random variables if one does not want to be limited to neutral attitudes toward risk.

In financial applications it may then seem desirable to use decision criteria which are setup in the purely subjective framework of Savage (1954), where the set X has no structure imposed upon. While Ghirardato et al. (2003) have designed "subjective mixtures" that should allow to transpose models developed in the AA framework into a fully subjective setting, most of the recent models have yet to be adapted. Hence the financial economist wishing to explore the consequences of decision-making under ambiguity has to model random variables in the AA framework.

The standard practice is to use purely subjective lotteries, that is acts that map states to degenerate distributions in the set X

E-mail address: eandre@em-lyon.com.

[1] See Guidolin and Rinaldi (2013) for a recent survey.

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of objective lotteries. This definition of *monetary acts* is found in major research articles (e.g. Section 3.6 of Maccheroni et al., 2006 henceforth abbreviated MMR06) and in reference finance textbooks (e.g. Section 2.5 in Föllmer and Schied, 2011). In financial applications we then restrict the choice set to a subset of the acts. At first, this does not seem to be an issue. One can even argue with David Kreps that, in the AA framework, we have "enrich[ed] the choice set with imaginary objects—in this case compound lotteries" to "make it easier to get the representation" but that "in most applications of this theory, [the] actual available choices will come only from F [the subset of purely subjective lotteries]" (Kreps, 1988, p. 101).

But this restriction to purely subjective acts is not without consequences. A striking example is found in the axiomatization of the multiplier preferences of Hansen and Sargent (2001) by Strzalecki (2011). The author proves that these are a specification of the variational preferences of MMR06 obtained by imposing Savage's P2 axiom, that is EU, to purely subjective lotteries. Indeed, "multiplier preferences rank purely subjective acts according to the EU criterion" hence "it is not possible to distinguish multiplier preferences from the EU preferences based on the preferences over purely subjective acts alone" (Strzalecki, 2011, note 10). Therefore, in a financial application where asset prices or returns are represented by monetary acts, an investor with such preferences would hold the same portfolio as an EU maximizer.

In this paper, we study another consequence of this restriction to purely subjective acts for models characterized by a set of priors $\mathscr{C}$. Given this set, it is possible, following Ghirardato et al. (2004), to identify the *crisp acts* whose "evaluation is not affected by the ambiguity the DM [decision maker] displays in the decision problem": their expected utility is the same, whichever of the priors in $\mathscr{C}$ is used. In multiple priors models of decision under ambiguity, the investor has to be indifferent between holding a constant act and non constant crisp act with the same expected utility. We show that this indifference, together with the standard definition of monetary acts in the Anscombe–Aumann framework, implies that the investor ignores the variance of some assets, a behavior which conflicts with the assumption on which modern portfolio theory has been built.

Is it then possible to exclude non constant crisp acts from financial settings ? In this paper we start by establishing a unique decomposition of the utility profiles of crisp acts into a constant and a non constant part, that we name *crisp fair gamble*. We then study the conditions for the exclusion of crisp fair gambles and what they mean in a financial setting. To this end, we establish some links between the geometry of the set of priors and the subspace of crisp fair gambles in the finite and the infinite dimensional cases. We start from the very general setting of the rational preferences proposed by CGMMS11. This setup is summarized in Section 2. The definition of the crisp fair gambles and the indifference property are in Section 3. The main mathematical results are in Section 4 where geometrical and topological properties of the set of priors and of the subspace of crisp fair gambles are proved in finite and infinite dimension. These properties allow to state necessary and sufficient conditions which, when imposed to the set of priors, ensure the non existence of crisp fair gambles. Finally Section 5 gives the link with the unambiguous acts and a necessary condition on the set of unambiguous events which has a direct consequence for financial settings.

2. The setup: rational preferences under ambiguity

We set this paper in the general multiple priors settings of the "Rational preferences under ambiguity" as they have been introduced by CGMMS11. This paper generalizes Ghirardato et al. (2004)'s results on the identification of a set of "relevant priors" and on the definition of unambiguous acts and events to most of the standard models of decision under ambiguity, such as the Maxmin Expected Utility (Gilboa and Schmeidler, 1989), the Choquet Expected Utility (Schmeidler, 1989), the Smooth Model (Klibanoff et al., 2005), the Variational preferences (MMR06), the Vector Expected Utility (Siniscalchi, 2009), the homothetic uncertainty averse preferences (Chateauneuf and Faro, 2009) or the Uncertainty Averse preferences (Cerreia-Vioglio et al., 2011b).

2.1. Setup and notations

We consider a state space S endowed with an algebra Σ, and X a convex subset of a vector space. Simple acts are Σ-measurable functions $f: S \to X$ such that $f(S)$ is finite. The set of all acts is denoted by $\mathcal{F}$. Given an $x \in X$, define $x \in \mathcal{F}$ to be the constant act such that $x(s) = x$ for all $s \in S$. With the usual slight abuse of notation, we can then identify X with the subset of constant acts in $\mathcal{F}$.

$B_0(\Sigma, I)$ is the space of simple Σ-measurable function on S with values in the interval $I \subset \mathbb{R}$. We write $B_0(\Sigma)$ instead of $B_0(\Sigma, \mathbb{R})$ for the space of finite linear combinations of characteristic functions of sets in Σ. $ba(\Sigma)$, $ba_1(\Sigma)$ and $ca_1(\Sigma)$ denote respectively the spaces of finitely additive measures, finitely additive probabilities and countably additive probabilities on Σ. These spaces are endowed with the total variation norm. $B(\Sigma)$ is the space of all uniform limits of the functions in $B_0(\Sigma)$. Endowed with the supnorm $\|\cdot\|_{sup}$ it is a Banach space whose topological dual is isometrically isomorphic to $ba(\Sigma)$. We will write the duality pairing as $\langle a, \mu\rangle = \int a\, \mathrm{d}\mu$, the hyperplane $\mathsf{H}_{\mu,\alpha} = \{a \mid \langle a, \mu\rangle = \alpha\}$ and the closed half space $\mathsf{H}^+_{\mu,\alpha} = \{a \mid \langle a, \mu\rangle \geq \alpha\}$. We denote by $\mathbf{1}_E$ the characteristic function of the event $E \in \Sigma$.

A functional $I: B_0(\Sigma, I) \to \mathbb{R}$ is *normalized* if $I(\alpha \mathbf{1}_S) = \alpha$ for all $\alpha \in I$, *monotonic* if $a \geq b \Rightarrow I(a) \geq I(b)$ and *continuous* if it is sup-norm continuous.

2.2. The preference

The DM preference is modeled by a binary relation $\succcurlyeq$ over $\mathcal{F}$. CGMMS11 consider a preference that satisfy a minimal set of four axioms: (i) *Weak Order*: $\succsim$ is non-trivial, complete and transitive on $\mathcal{F}$, (ii) *Monotonic*: if $f, g \in \mathcal{F}$ and $f(\omega) \succsim g(\omega)$ for all $\omega \in \Omega$, then $f \succsim g$, (iii) *Risk Independence*: if $x, y, z \in X$ and $\lambda \in (0, 1]$, then $x \succ y$ implies $\lambda x + (1-\lambda)z \succ \lambda y + (1-\lambda)z$, (iv) *Archimedean*: If $f, g, h \in \mathcal{F}$ and $f \succ g \succ h$, then there are $\alpha, \beta \in (0, 1)$ such that $\alpha f + (1-\alpha)h \succ g \succ \beta f + (1-\beta)h$. This preference is Monotonic, Bernoullian (it admits a "Bernoulli utility index", that is an affine utility function $u: X \to \mathbb{R}$ that represents the restriction of $\succcurlyeq$ to X) and Archimedean, hence it has been dubbed the MBA preference.

Proposition 1 (*CGMMS11*)**.** *A preference relation $\succcurlyeq$ satisfies the MBA axioms if and only if there exist a non-constant, affine function $u: X \to \mathbb{R}$ and a normalized, monotonic, continuous functional $I: B_0(\Sigma, u(X)) \to \mathbb{R}$ such that for each $f, g \in \mathcal{F}$,*

$$f \succcurlyeq g \iff I(u \circ f) \geq I(u \circ g). \tag{1}$$

The function I depends on the choice of the utility function u but is then uniquely determined by this choice. The domain of definition of I is $B_0(\Sigma, u(X))$, the set of utility profiles, that is $u \circ \mathcal{F} \stackrel{\text{def}}{=} \{u \circ f \mid f \in \mathcal{F}\}$. While the results of this paper still hold if $u(X)$ is a proper subset of $\mathbb{R}$, given our focus on monetary acts, we assume from now on that $u(X) = \mathbb{R}$.

2.3. The revealed unambiguous preference and crisp acts

The incomplete revealed unambiguous preference relation $\succcurlyeq^*$ collects the pairs of acts whose rankings are unaffected by