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A fixed-point approach to the limit load analysis of multibody structures with Coulomb friction

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ABSTRACT

In this paper, it is shown that, for planar systems formed from linear-elastic bodies in non-penetrative contact with Coulomb friction, the limit load can be approximated efficiently by the limit loads corresponding to a sequence of contact problems with given friction, via a finite element fixed-point approach. For the auxiliary problem with given friction, a splitting of this problem into two subproblems, one with prescribed normal contact stresses and given friction and one with prescribed tangential contact stresses, is employed. Numerical results for masonry block structures subjected to in-plane loading illustrate the predictive capabilities of this approach.

1. Introduction

Contact problems with friction are central to the modelling and analysis of many structural systems (e.g. in architecture and constructions, biomechanics, automotive and aerospace engineering, robotics, computer graphics, etc.) and the numerical treatment of these problems poses many theoretical and computational challenges (see e.g. [32,48,1]). The complexity of contact problems modelling structural systems is generally associated with the detection of contacts and openings and the resolution of non-linear equations for contact (see [12,20,27,33,22,13]). However, a whole new level of complexity arises when a system is subjected to loading that can lead to loss of equilibrium and the corresponding limit load needs to be evaluated as well. In this case, the problem is, on the one hand, to determine the limit value of the load, such that the system acted upon by loads smaller than the limit load will not collapse (the static principle), and on the other hand, to predict a possible collapse mode for the structure under the limit load (the kinematic principle) (see [8,23,25,42,47]).

For systems formed from linear-elastic bodies in mutual non-penetrative contact with Tresca (given) friction, the limiting tangential force associated with sliding at the contact zone is independent of the normal compressive force, and the static and kinematic principles of limit load analysis take the form of two dual problems in infinite dimensional convex programming (see [8, Section 6.4]). For planar systems, these problems are analysed in Mihai and Ainsworth [41], where a unified theory for the existence of a solution is established, in the framework of variational inequalities, and two finite element approaches for the computation of guaranteed lower and upper bounds on the unique limit load are devised. In the first approach, the static limit load problem reduces to a linear program (LP) in the divergence-free space, from which a lower (safe) bound for the limit load and the corresponding stress distribution are obtained. In the second approach, the kinematic limit load problem is expressed as a mixed linear complementarity problem (MLCP) from which an upper (unsafe) bound for the limit load and the collapse mode are determined.

In contact mechanics, the challenge is to apply models for which parameters can be easily identified from experiments and which lend themselves to a reliable numerical treatment. The contact model with given friction may be appropriate for certain applications (see e.g. [19,39]) and can be represented as a linear or convex programming problem, for which robust and efficient solution procedures are available (see e.g. [17,49,44]). However, for many practical problems, the Coulomb condition that a tangential force less than a critical value proportional to the normal compressive force will not cause sliding (cf. [9,24]) is more realistic. Unfortunately, the Coulomb friction condition introduces non-linearity and non-convexity in the optimization problem, which render it very difficult to solve.

The limit load analysis of elasto-plastic systems with Coulomb friction was considered, perhaps for the first time, in Drucker [11], where analytical lower and upper bounds for the limit load were established by approximating the Coulomb friction model by models with given friction. As a result, a lower bound was

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given by the limit load for frictionless contact, while an upper bound was found by solving the limit load problem for contact without relative sliding at interfaces. The first numerical approach for the limit load problem of multibody systems formulated as an LP was introduced in Livesley [34], where a computational procedure previously developed for the limit load analysis of plastic frames was extended to the analysis of rigid-block systems with frictional contact interfaces. More recently, numerical procedures for the limit load analysis of systems formed from rigid-blocks in frictional contact were proposed in Fishwick [16], Baggio and Trovalusci [6], Ferris and Tin-Loi [15], Gilbert et al. [18]. For large-scale systems, in [15], the limit load problem for discrete models with non-associative friction is expressed as a mathematical program with equilibrium constraints (MPEC) (for applications, see also Orduña and Lorenço [45]), while in [18], this limit load problem is approximated by a sequence of problems with successively modified associative friction. Extensions to finite element models of systems formed from linear-elastic blocks were proposed in Maier and Nappi [38] and Boothby and Brown [7]. Recently, an efficient finite element procedure for the limit load assessment of elasto-plastic structures with general piecewise linear yield conditions, in discrete framework, is discussed in Ardito et al. [5].

While the study of discrete limit load problems is motivated by the rapid development of robust optimization techniques on which it strongly relies, considerably less attention has been paid to the associated continuous problems. The main objective of this paper is to extend the analysis presented in [41] to planar systems with Coulomb friction, by combining the limit load principles with well-known techniques for the representation of multibody systems with contact constraints. At continuous level, for a system of linear-elastic bodies in non-penetrative contact with Coulomb friction, when the coefficient of friction is sufficiently small, an equilibrium solution can be obtained by a fixed-point approach involving contact problems with given friction (cf. [43,30,31]). For an effective numerical realisation of the auxiliary problem with given friction, at each fixed-point iteration, a splitting of this problem into two subproblems, one with prescribed normal contact stresses and given friction and one with prescribed tangential contact stresses, can be employed (cf. [46,21,10]). The fixed-point approach combined with the splitting method has been successfully applied to the simulation of large-scale multibody structures in Refs. [3,4,40]. In this paper, the fixed-point iteration and the splitting procedure are extended to the numerical limit load analysis of multibody systems with Coulomb friction. In Section 2, the limit load problem for multibody systems with frictional contact constraints is introduced at infinite dimensional level. In Section 3, the fixed-point algorithm for the numerical solution of the limit load problem for systems with Coulomb friction, discretised by the finite element method, is described and analysed. This is coupled, in Section 4, with a splitting procedure for the auxiliary problem with given friction. The resulting algorithm is presented in an algebraic form suitable for computer implementation in Section 5. In Section 6, numerical results for masonry block structures subjected to in-plane loading illustrate the predictive capabilities of the fixed-point approach. Conclusions are drawn in Section 7. Additional technical details are provided in the two appendices at the end of the paper.

2. The limit load problem

Let $\Omega = \Omega_1 \cup \cdots \cup \Omega_{m_s} \subset \mathbb{R}^2$, $m_s \geqslant 2$, represent the domain occupied by a system formed from linear elastic bodies in non-penetrative contact with friction, where every Ω_s is a simply connected polygonal domain filled by a single body, $s = 1, \ldots, m_s$. The global boundary $\Gamma = \partial\Omega_1 \cup \cdots \cup \partial\Omega_{m_s}$ is partitioned as $\Gamma = \Gamma_C \cup \Gamma_E$, where $\Gamma_C \neq \emptyset$ is the potential contact zone consisting of the interfaces between bodies, and $\Gamma_E = \Gamma \setminus \Gamma_C$ is the exterior part of the boundary.

2.1. Boundary conditions

The exterior boundary is partitioned as $\Gamma_E = \Gamma_D \cup \Gamma_S \cup \Gamma_B$, where:

- On Γ_D the system is fixed:

$$u_N = 0 \quad \text{and} \quad u_T = 0.$$

- On Γ_S a simple support is assumed:

$$u_N = 0 \quad \text{and} \quad \sigma_T = 0.$$

- On $\Gamma_B \neq \emptyset$ boundary tractions are acting:

$$\sigma_N = F_N^B \quad \text{and} \quad \sigma_T = F_T^B.$$

2.2. Contact conditions

On the contact zone Γ_C:

- The non-penetrative contact is modelled by the conditions:

$$[u_N] \geqslant 0, \quad \sigma_N \leqslant 0, \quad [u_N]\sigma_N = 0. \tag{2.1}$$

- The frictional contact is governed by the classical law:

$$[u_T] = 0 \ \Rightarrow \ |\sigma_T| \leqslant \mathcal{F}G,$$
$$[u_T] \neq 0 \ \Rightarrow \ \sigma_T = -\mathcal{F}G\frac{[u_T]}{|[u_T]|}. \tag{2.2}$$

In the above relations, the indices N and T indicate the normal and tangential directions, respectively, which are given an arbitrarily unique value on every common edge between two bodies, u_N, u_T are the normal and tangential displacements, σ_N, σ_T are the normal and tangential stresses, [] represents the jump across a contact edge, $\mathcal{F} > 0$ is the coefficient of friction, and $G \geqslant 0$ is as follows: for contact with Coulomb friction $G = |\sigma_N|$, for contact with Tresca friction $G > 0$ is prescribed, and for frictionless contact $G = 0$.

The friction law (2.2) can be expressed equivalently as follows (see e.g. [22, p. 377]):

$$|\sigma_T| \leqslant \mathcal{F}G, \quad [u_T](|\sigma_T| - \mathcal{F}G) = 0, \quad [u_T]\sigma_T \leqslant 0 \tag{2.3}$$

or in complementarity (Kuhn–Tucker) form (see e.g. [48, p. 83]):

$$|\sigma_T| \leqslant \mathcal{F}G, \quad v_T(|\sigma_T| - \mathcal{F}G) = 0, \quad v_T \leqslant 0, \tag{2.4}$$

where $v_T = [u_T]n_T$ and $n_T = \sigma_T/|\sigma_T|$. Clearly the relations (2.3) and (2.4) are equivalent and express the fact that the absolute value of the tangential contact stress is less than or equal to $\mathcal{F}G$ and, if this value is attained, then tangential slip can occur in the direction opposite to that of the frictional force. In the subsequent analysis, the formulation (2.3) is employed, while the complementarity form (2.4) will be useful for the numerical computations. For completeness, a detailed proof of the equivalence between (2.2) and (2.3) is given in Appendix A.

2.3. Loading conditions

The given system is subjected initially to a dead load (preload) induced by the volume force $\mathbf{f}^D$ over Ω and the boundary tractions $\mathbf{F}^B$ on $\Gamma_B \subset \Gamma_E$, $\Gamma_B \neq \emptyset$. Assuming that an optimal solution is already known for the contact problem representing the system under the preload, when additional (live) body forces $\lambda\mathbf{f}^L$ and boundary tractions $\lambda\mathbf{F}^L$ are applied, the problem is to determine the range