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Minimal extending sets in tournaments

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HIGHLIGHTS

- ME satisfies idempotency, irregularity, and inclusion in the iterated Banks set.
- ME violates monotonicity, stability, and computational tractability.
- Concrete counterexamples for monotonicity and stability remain unknown.

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ABSTRACT

Tournament solutions play an important role within social choice theory and the mathematical social sciences at large. In 2011, Brandt proposed a new tournament solution called the *minimal extending set* (*ME*) and an associated graph-theoretic conjecture. If the conjecture had been true, *ME* would have satisfied a number of desirable properties that are usually considered in the literature on tournament solutions. However, in 2013, the existence of an enormous counter-example to the conjecture was shown using a non-constructive proof. This left open which of the properties are actually satisfied by *ME*. It turns out that *ME* satisfies idempotency, irregularity, and inclusion in the iterated Banks set (and hence the Banks set, the uncovered set, and the top cycle). Most of the other standard properties (including monotonicity, stability, and computational tractability) are violated, but have been shown to hold for all tournaments on up to 12 alternatives and all random tournaments encountered in computer experiments.

1. Introduction

Many problems in the mathematical social sciences can be addressed using tournament solutions, i.e., functions that associate with each connex and asymmetric relation on a set of alternatives a non-empty subset of the alternatives. Tournament solutions are most prevalent in social choice theory, where the binary relation is typically assumed to be given by the simple majority rule (e.g., Moulin, 1986; Laslier, 1997). Other application areas include multi-criteria decision analysis (e.g., Arrow and Raynaud, 1986; Bouyssou et al., 2006), zero-sum games (e.g., Fisher and Ryan, 1995; Laffond et al., 1993; Duggan and Le Breton, 1996), and coalitional games (e.g., Brandt and Harrenstein, 2010).

Examples of well-studied tournament solutions are the Copeland set, the uncovered set, and the Banks set. A common benchmark for tournament solutions is which desirable properties they satisfy (see, e.g., Laslier, 1997; Brandt et al., 2016, for an overview of tournament solutions and their axiomatic properties).

In 2011, Brandt (2011) proposed a new tournament solution called the *minimal extending set* (*ME*) and an associated graph-theoretic conjecture, which weakens a 20-year-old conjecture by Schwartz (1990). Brandt's conjecture is closely linked to the axiomatic properties of *ME* in the sense that if the conjecture had held, *ME* would have satisfied virtually all desirable properties that are usually considered in the literature on tournament solutions. In particular, it would have been the only tournament solution known to simultaneously satisfy stability and irregularity. In 2013, however, the existence of a counter-example with about 10^{104} alternatives was shown.[1] The proof is non-constructive and uses the probabilistic method (Brandt et al., 2013). This counter-example also disproves Schwartz's conjecture and implies that the *tournament equilibrium set* – a tournament solution proposed by Schwartz (1990) – violates most desirable axiomatic properties.[2]

[1] The bound is $\binom{2^{15}}{30} < 10^{104}$. The weaker bound of 10^{136} mentioned by Brandt et al. (2013) stems from the estimate $\binom{2^{15}}{30} < 2^{15^{30}}$.

[2] A significantly smaller counter-example for Schwartz's conjecture consisting of only 24 alternatives was subsequently found by Brandt and Seedig (2013). However, this counter-example does not constitute a counter-example to Brandt's conjecture.

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This left open which of the properties are actually satisfied by *ME*. In this paper, we resolve these open questions. In particular, we show that *ME* fails to satisfy monotonicity, stability, and computational tractability while it does satisfy a strengthening of idempotency, irregularity, and inclusion in the (iterated) Banks set.[3] Our negative theorems for monotonicity and stability are based on the non-constructive existence proof by Brandt et al. (2013). Concrete tournaments for which *ME* violates any of these properties therefore remain unknown.

2. Preliminaries

A *tournament* T is a pair $(A, \succ)$, where A is a finite set of alternatives and $\succ$ a binary relation on A, usually referred to as the *dominance relation*, that is both asymmetric ($a \succ b$ implies not $b \succ a$) and connex ($a \neq b$ implies $a \succ b$ or $b \succ a$). Thus, the dominance relation is generally irreflexive (not $a \succ a$). Intuitively, $a \succ b$ signifies that alternative a is preferable to alternative b and we denote this by an edge from a to b in our figures. The dominance relation can be extended to sets of alternatives by writing $A \succ B$ when $a \succ b$ for all $a \in A$ and $b \in B$. We also write $a \succ B$ for $\{a\} \succ B$. Moreover, for a subset of alternatives $B \subseteq A$, we will sometimes consider the restriction of the dominance relation $\succ_B = \succ \cap (B \times B)$ and write $T|_B$ for $(B, \succ_B)$. The *order* $|T|$ of a tournament $T = (A, \succ)$ refers to the cardinality of A. The set of all linear orders on some set A is denoted by $\mathcal{L}(A)$. Define the set of all transitive subsets of a tournament T as $\mathcal{B}_T = \{Q \subseteq A : \succ_Q \in \mathcal{L}(Q)\}$ whereas $\mathcal{B}_T(a) = \{Q \in \mathcal{B}_T : a \succ Q\}$ denotes the set of all transitive subsets that a dominates. In such a case, a *extends* Q, implying $Q \cup \{a\} \in \mathcal{B}_T$.

A *tournament solution* is a function S that maps a tournament $T = (A, \succ)$ to a nonempty subset of its alternatives. We write $S(B)$ instead of $S(T|_B)$ whenever the tournament T is clear from the context.

Choosing from a *transitive* tournament is straightforward because every transitive tournament – and all of its subtournaments – possess a unique maximal element. In other words, the core of the problem of choosing from a tournament is the potential intransitivity of the dominance relation. Clearly, every tournament contains transitive subtournaments. For example, all subtournaments of order one or two are trivially transitive. Based on these observations, it seems natural to consider *inclusion-maximal* transitive subtournaments and collect their maximal elements in order to define a tournament solution. This tournament solution is known as the *Banks set*.[4]

Formally, the Banks set $BA(T)$ of a tournament is defined as

$$BA(T) = \{a \in A : \exists B \in \mathcal{B}_T(a) \text{ such that } \nexists b \colon b \succ B \cup \{a\}\}.$$

In many cases, the Banks set contains all alternatives of a tournament. Since there are tournaments T for which $BA(BA(T)) \subsetneq BA(T)$, one can define a series of more discriminating tournament solutions by letting $BA^1(T) = BA(T)$ and $BA^k = BA(BA^{k-1}(T))$ for all $k > 1$. The *iterated Banks set* $BA^\infty(T)$ of a tournament T is then defined as

$$BA^\infty(T) = \bigcap_{k \in \mathbb{N}} BA^k(T).$$

Due to the finiteness of T, $BA^\infty(T) = BA^{|T|}(T)$, and BA^∞ is a well-defined tournament solution.

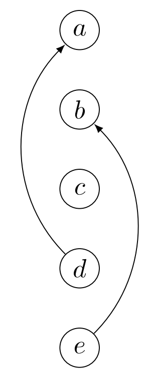

Fig. 1. In this tournament, $ME(T) = \{a, b, d\}$ whereas $BA(T) = \{a, b, c, d\}$. Omitted edges point downwards.

Generalizing an idea by Dutta (1988), which in turn is based on the well-established notion of von-Neumann–Morgenstern stable sets in cooperative game theory, Brandt (2011) proposed another method for refining a tournament solution S by defining minimal sets that satisfy a natural stability criterion with respect to S.[5] A subset of alternatives $B \subseteq A$ is called *S-stable* for tournament solution S if

$$a \notin S(B \cup \{a\}) \quad \text{for all } a \in A \setminus B.$$

Since $S(B \cup \{a\}) = \{a\}$ if $B = \emptyset$, it follows that S-stable sets can never be empty. It has turned out that BA-stable sets, so-called *extending sets*, are of particular interest because they are strongly related to Schwartz's tournament equilibrium set and because they can be used to define a tournament solution that potentially satisfies a number of desirable properties. An extending set is *inclusion-minimal* if it does not contain another extending set. Since the number of alternatives is finite, inclusion-minimal extending sets are guaranteed to exist. The union of all inclusion-minimal extending sets defines the tournament solution *ME* (Brandt, 2011), i.e.,

$$ME(T) = \bigcup \{B : B \text{ is } BA\text{-stable and no } C \subsetneq B \text{ is } BA\text{-stable}\}.$$

Example 1. Consider the tournament T in Fig. 1. It is easy to verify that the maximal transitive sets in T are $\{a, b, c\}$, $\{a, e, b\}$, $\{a, c, e\}$, $\{b, c, d\}$, $\{c, d, e\}$, and $\{d, a, e\}$. $\{e, b\}$ (the only nontrivial transitive subset with e as maximal element) is extended by a. Therefore, we have $BA(T) = \{a, b, c, d\}$.

We claim that $ME(T) = \{a, b, d\}$. To this end, let B be any extending set of T. Assume that $a \notin B$. Since B is non-empty and stable with respect to a, it must be the case that $d \in B$. Then, b has to be contained in B as well because no alternative could extend $\{b, d\}$. But then B cannot be stable with respect to a as there exists no alternative that could extend $\{a, b\}$. Therefore, $a \in B$ and immediately $d \in B$ (as nothing could extend $\{d, a\}$) and $b \in B$ (as nothing could extend $\{b, d\}$). It turns out that $\{a, b, d\}$ is already an extending set because $c \notin BA\{a, b, c, d\} = \{a, b, d\}$ and $e \notin BA\{a, b, d, e\} = \{a, b, d\}$. So, $\{a, b, d\}$ is the unique minimal extending set of T.

Note that $ME(T)$ is strictly contained in $BA(T)$. Tournament T is the smallest tournament for which this is the case (Brandt et al., 2015). For this particular tournament, $ME(T)$ and $BA^\infty(T)$ coincide.

We will show in Section 4.3 that

$$ME(T) \subseteq BA^\infty(T) \subseteq BA(T)$$

holds for all tournaments T and both inclusions may be strict.[6]

[3] Previously, the two statements on computational tractability and inclusion in the Banks set were only known to hold if the (now disproved) conjecture had been true.

[4] Banks's original motivation was slightly different as his aim was to characterize the set of outcomes under sophisticated voting in the amendment procedure (Banks, 1985).

[5] A well-known example is the *minimal covering set*, which is the unique minimal set that is stable with respect to the uncovered set (Dutta, 1988).

[6] An analogous inclusion chain is known for the uncovered set, the iterated uncovered set, and the minimal covering set (see, e.g., Laslier, 1997).