Mathematical Social Sciences 87 (2017) 85–91

Contents lists available at ScienceDirect

Mathematical Social Sciences

journal homepage: www.elsevier.com/locate/econbase

A simple method for generalized sequential compound options pricing

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HIGHLIGHTS

- We derive and generalize a mathematical expectation related to multivariate normal variables.
- The presented mathematical expectations are very useful for many types of options pricing.
- We propose a novel proof for sequential compound options pricing formula in the diffusion model.
- Analytic pricing formula for sequential compound options in the jump-diffusion model is obtained.

ARTICLE INFO

Article history:
Received 9 May 2015
Received in revised form
19 June 2016
Accepted 3 March 2017
Available online 10 March 2017

ABSTRACT

This paper presents a new and simple method to derive the pricing formula for generalized sequential compound options (SCOs). Multi-fold generalized SCOs are defined as compound options on (compound) options, where the call/put property of each fold can be arbitrarily assigned. To obtain the analytic pricing formula for n-fold generalized SCOs, we prove and generalize a mathematical expectation related to multivariate normal variables, which are potentially very useful in pricing many types of option. Subsequently, with the help of the proven conclusions, the n-fold generalized SCOs pricing formulas for the diffusion model and the log-normal jump-diffusion model are derived. Finally, some possible computational methods for the calculation of SCOs price are presented.

1. Introduction

Compound options are options that have other options as underlying assets. Geske (1979) first presented the original closed form pricing formula for 2-fold compound options using a partial differential equation method and this paper set precedent for later works. Since then, some scholars have extended the pricing model and proposed some new pricing methods for compound options. For example, Agliardi and Agliardi (2003) generalized the results to 2-fold compound calls with time-dependent parameters. Lajeri-Chaherli (2002) used the martingale approach and the expectation of truncated bivariate normal variables to prove the pricing formula for 2-fold compound options. Fouque and Han (2005) proposed a perturbation approximation to compute the prices of compound options. Gukhal (2004) derived analytical valuation formulas for 2-fold compound options when the underlying value follows a log-normal jump-diffusion process. Chiarella and Kang (2011) presented the evaluation of American compound option prices under stochastic volatility and stochastic interest rates. Griebsch (2013) evaluated European compound option prices under stochastic volatility using Fourier transform techniques. Chiarella et al. (2014) provided an in-depth analysis of several structurally different methods to numerically evaluate European compound option prices under Heston's stochastic volatility dynamics. In addition, some examples in the literature have developed 2-fold to n-fold compound options, and studied the pricing and sensitivity analysis for the n-fold compound options. Specific multi-fold compound option pricing formulas have been proposed by Geske and Johnson (1984) and Carr (1988), while the pricing formula of sequential compound call options (SCCs) was proved by Thomassen and Van Wouwe (2001) and Chen (2003). Agliardi and Agliardi (2005) derived the closed-form solution for multi-fold compound calls when volatility and interest rate vary with time. Andergassen and Sereno (2012) presented the n-fold SCCs pricing formula in the jump-diffusion model. Lee et al. (2008) were the first to extend the sequential compound call options to generalized

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http://dx.doi.org/10.1016/j.mathsocsci.2017.03.001
0165-4896/

sequential compound options (SCOs) and presented the pricing formula and sensitivities for sequential compound options. The multi-fold sequential compound options proposed in Lee et al. (2008) are defined as compound options on (compound) options where the call/put property of each fold can be arbitrarily assigned.

Compound option is widely employed in the field of financial derivatives pricing, for instance, American put options (Geske and Johnson, 1984; Gukhal, 2004), sequential exchange options (Carr, 1988), as well as sequential American exchange property options (Paxson, 2007). In addition to the pricing of financial derivatives, compound option theory is extensively used in the real option field. Examples include project valuation (Cassimon et al., 2011, 2004; Pennings and Sereno, 2011; Nigro et al., 2014; Huang and Pi, 2009), decision-making (Park et al., 2013), and banking crisis (Maltritz, 2010; Eichler et al., 2011). Compound options also play an important theoretical role in completing markets (Nachman, 1989; Arditti and John, 1980; Green and Jarrow, 1987).

The motivation of this paper are as follows. First, the wide deployment of financial derivatives in the real options field have revealed that the limitations of the current compound option methodology which is based on 2-fold compound options, or the results concerning multi-fold compound options so far have focused only on sequential compound calls. In the real world, many cases can be expressed in terms of options, such as expansion, contraction, shutting down, abandon, switch, and/or growth, so the generalized SCOs may be a very useful instrument to treat many real-world cases. An example quoted in Lee et al. (2008) is that the effect of revenue guarantee in a build–operate–transfer (BOT) project of utility construction can be evaluated by SCOs. Second, although Lee et al. (2008) have presented the analytic pricing formula for generalized sequential compound options, it is not hard to find that even in the most simple model, i.e., diffusion model, the derivation for analytic pricing formula is quite complex. We can imagine that it will be a very difficult thing to derive the pricing formula for generalized sequential compound options in complex models, such as jump-diffusion model and Levy model. Therefore, we should seek some new and simple methods to obtain the pricing formula for generalized sequential compound options.

Undoubtedly, the sophisticated structure of sequential compound options necessarily leads to some difficulties in the derivation of the pricing formula. As far as we know, only Lee et al. (2008) have derived the pricing formula and sensitivity analysis for generalized sequential compound options in the diffusion model with the deterministic time-dependent parameters. With the help of the relationship between the $(k-1)$ and k-variate normal integrals, Lee et al. (2008) used the risk-neutral pricing method to derive the pricing formula for multi-fold SCOs by induction. In this paper, in order to obtain the analytic pricing formula for n-fold generalized SCOs, we prove and generalize a mathematical expectation about multivariate normal variables, by means of the conclusions we derive the n-fold sequential compound options pricing formula under the assumption that the underlying asset price follows a diffusion process and jump-diffusion process using the risk-neutral method, respectively. The main contributions of this paper are as follows. First, we generalize the mathematical expectation related to multivariate normal variables which was firstly given in Wang and He (2016). Second, we present a novel and alternative proof for the generalized sequential compound options pricing formula. Finally, this paper present a pricing formula for generalized SCOs under the assumption that the underlying asset price follows a log-normal jump-diffusion process. Therefore, the present paper can be treated as a substantial generalization of Lee et al. (2008).

The rest of this paper is structured as follows. A mathematical expectation about multivariate normal variables and its generalization are proved in Section 2. Sections 3 and 4 present the SCOs pricing formula in the diffusion and jump-diffusion models, respectively. Some possible computational methods for SCOs price are provided in Section 5. Conclusions are stated in Section 6.

2. A mathematical expectation related to multivariate normal variables

In order to derive the mathematical expectation about multivariate normal variables, we give the succeeding Lemma 1 from Wang and He (2016).

Lemma 1. *Let $\Sigma = (\omega_{ij})_{n\times n}$ be a symmetric and positive definite matrix*

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \cdots & \rho_{1n}\sigma_1\sigma_n \\ \rho_{12}\sigma_1\sigma_2 & \sigma_2^2 & \cdots & \rho_{2n}\sigma_2\sigma_n \\ \cdots & \cdots & \cdots & \cdots \\ \rho_{1n}\sigma_1\sigma_n & \rho_{2n}\sigma_2\sigma_n & \cdots & \sigma_n^2 \end{pmatrix}. \tag{1}$$

Then for any $x_1, x_2, \ldots, x_n \in R$,

$$\begin{aligned}&(x_1-\rho_{1n}\sigma_1\sigma_n, x_2-\rho_{2n}\sigma_2\sigma_n, \ldots, x_n-\sigma_n^2)\Sigma^{-1}\\ &\quad\times(x_1-\rho_{1n}\sigma_1\sigma_n, x_2-\rho_{2n}\sigma_2\sigma_n, \ldots, x_n-\sigma_n^2)'\\ &= (x_1, x_2, \ldots, x_n)\Sigma^{-1}(x_1, x_2, \ldots, x_n)' - 2x_n + \sigma_n^2\end{aligned} \tag{2}$$

Lemma 2. *Let n-variate normal random vector $\boldsymbol{X} \sim N(\boldsymbol{0}, \Sigma)$, where*

$$\boldsymbol{X} = (X_1, X_2, \ldots, X_n)' \tag{3}$$

and Σ is the covariance matrix of $\boldsymbol{X}$, which is denoted as (1). That is, the correlation matrix of $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)'$ is

$$A = \begin{pmatrix} 1 & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{12} & 1 & \cdots & \rho_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \rho_{1n} & \rho_{2n} & \cdots & 1 \end{pmatrix}. \tag{4}$$

Then for any constant $c_1, c_2, \ldots, c_n$, there is:

$$\begin{aligned}&E(e^{X_n} I_{\{X_1 \wedge_1 c_1, X_2 \wedge_2 c_2, \ldots, X_n \wedge_n c_n\}})\\ &= e^{\frac{\sigma_n^2}{2}} N_n\Bigg(\triangle_1 \frac{-c_1+\rho_{1n}\sigma_1\sigma_n}{\sigma_1}, \triangle_2 \frac{-c_2+\rho_{2n}\sigma_2\sigma_n}{\sigma_2},\\ &\qquad \ldots, \triangle_n \frac{-c_n+\sigma_n^2}{\sigma_n}; Q\Bigg)\end{aligned} \tag{5}$$

where $I(\cdot)$ is an indicator function. The notation Λ_i denotes "$\geq$" or "$\leq$", and $\triangle_i = 1$ if Λ_i is "$\geq$"; $\triangle_i = -1$ if Λ_i is "$\leq$", $i = 1, 2, \ldots, n$.

$$\begin{aligned}N_n(x_1, \ldots, x_n; Q) &= \frac{1}{(2\pi)^{\frac{n}{2}} \mid Q \mid^{\frac{1}{2}}}\\ &\times \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} \exp\left\{-\frac{1}{2}(x_1, \ldots, x_n)Q^{-1}(x_1, \ldots, x_n)'\right\}\\ &\quad dx_1 \cdots dx_n\end{aligned} \tag{6}$$

is the n-variate standard normal distribution function with correlation matrix,

$$Q = \begin{pmatrix} 1 & \triangle_{12}\rho_{12} & \cdots & \triangle_{1n}\rho_{1n} \\ \triangle_{12}\rho_{12} & 1 & \cdots & \triangle_{2n}\rho_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \triangle_{1n}\rho_{1n} & \triangle_{2n}\rho_{2n} & \cdots & 1 \end{pmatrix} \tag{7}$$

and $\triangle_{ij} = \triangle_i \times \triangle_j$, $i, j = 1, 2, \ldots, n$.

Proof. Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)'$, $\boldsymbol{b} = (\rho_{1n}\sigma_1\sigma_n, \rho_{2n}\sigma_2\sigma_n, \ldots, \sigma_n^2)'$, $D = \{\boldsymbol{x} | x_1 \Lambda_1 c_1, x_2 \Lambda_2 c_2, \ldots, x_n \Lambda_n c_n\}$. By the expectation of